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# METAS VNA Tools - Math Reference V3.0.1

Michael Wollensack & Johannes Hoffmann

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### 1 Introduction

This document describes the computation of the uncertainties of coaxial S-parameter measurements, see [1]. This is a rather complicated task as all treated quantities are complex and the required operations are numerous.

Up to now a common way to handle uncertainties of vector network analyzer (VNA) measurements is the ripple technique described in [2]. The ripple technique uses precision airlines and other physical standards to extract the residual errors of a calibrated VNA system. It is based on the assumption that the precision transmission line has zero reflection. However, the systematic reflection coefficients of connectors make this assumption invalid, see [3]. This makes that the ripple method is unsuitable for computing the uncertainty of very accurate measurements. A more sophisticated computation of uncertainties has been implemented in StatistiCAL [4]. It relies on predefined uncertainties in standards and raw data. Thus it can capture certain imperfections of the VNA and the used standards. On the other hand it has neither a clear Bayesian nor frequentist concept for the imperfections of the VNA and does not provide means for producing a detailed uncertainty budget.

The present document describes in a first part the measurement model. A very well known measurement model for VNAs is described in [5]. However the model used here is slightly different. It is a  $N$ -port model and it has a more detailed uncertainty mechanism than [5]. The second part is the propagation of uncertainties through this measurement model. This methodology is described in the Guide to the Expression of Uncertainty in Measurement (GUM) [6], [7]. METAS UncLib [8], [9] is used for the linear propagation of uncertainties. The result is not only an uncertainty region but a list of uncertainty contributions with correlations. Thus the uncertainties can be propagated into eventual post-processing steps. A short description of METAS VNA Tools can be found in [10].



## 2 VNA Measurement Model

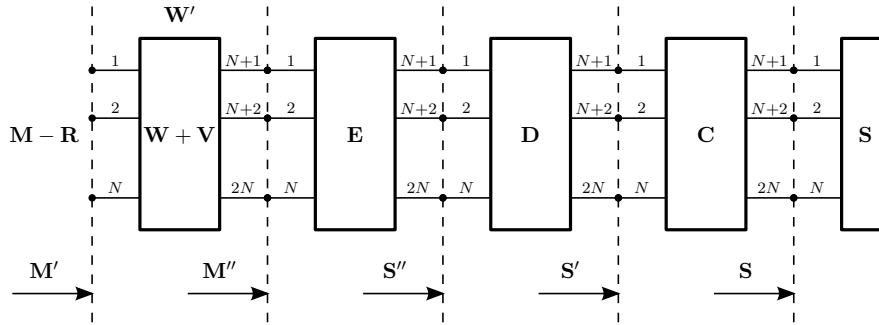


Figure 1: VNA Measurement Model

The following equation describes the in METAS VNA Tools used  $N$ -port VNA measurement model.

$$\mathbf{M}^{(i)} = \mathbf{R}^{(i)} + \left[ \left( \mathbf{W} + \mathbf{V}^{(i)} \right) \oplus \left[ \mathbf{E} \oplus \left[ \mathbf{D}^{(i)} \oplus \left[ \mathbf{C}^{(i)} \oplus \mathbf{S}^{(i)} \right] \right] \right] \right] \quad (1)$$

All bold variables are S-parameter matrices.  $\mathbf{M}$ ,  $\mathbf{R}$  and  $\mathbf{S}$  are  $N$ -ports, the other bold variables are  $2N$ -ports and  $i$  is the measurement index.

$\mathbf{M}$  denotes the raw data measured by the VNA.

$\mathbf{R}$  denotes the noise and linearity influences.

$\mathbf{W}$  denotes the switch terms.

$\mathbf{V}$  denotes the drift of the switch terms.

$\mathbf{E}$  denotes the calibration error terms.

$\mathbf{D}$  denotes the drift of the calibration error terms.

$\mathbf{C}$  denotes the cable stability, connector repeatability, switchable unit repeatability and DUT uncertainty influences.

$\mathbf{S}$  denotes the error corrected data or the calibration kit standard definitions.

$\mathbf{M}$ ,  $\mathbf{R}$ ,  $\mathbf{V}$  and  $\mathbf{D}$  change from measurement to measurement.  $\mathbf{W}$  and  $\mathbf{E}$  are constant during an entire calibration.  $\mathbf{C}$  changes for every new connection or cable movement.  $\mathbf{S}$  changes if a new device is connected.

The operator  $\oplus$  denotes the cascading of two S-parameter sets, as described in appendix A.1.1.

The inverse function of equation 1 can be used for error correction.

$$\mathbf{S}^{(i)} = \left[ \left[ \left[ \left( \mathbf{M}^{(i)} - \mathbf{R}^{(i)} \right) \ominus \left( \mathbf{W} + \mathbf{V}^{(i)} \right) \right] \ominus \mathbf{E} \right] \ominus \mathbf{D}^{(i)} \right] \ominus \mathbf{C}^{(i)} \quad (2)$$

The operator  $\ominus$  denotes the decascading of two S-parameter sets, as described in appendix A.1.1.





### 3 VNA Generic Calibration Model

The Generic VNA Model is used for the following calibration types: One Port, GSOLT, QSOLT [11], Unknown Thru [12], TRL [13], LRRM [14], Juroshek [15] and LHKM [16], [17]. If an  $N$ -port VNA has  $2N$  receivers instead of  $N + 1$  receivers then the switch terms can be measured directly, see [18]. For one-port measurements the switch terms and the associated drift can be set to zero.

#### 3.1 One Port Calibration

##### 3.1.1 Reflection Error Terms

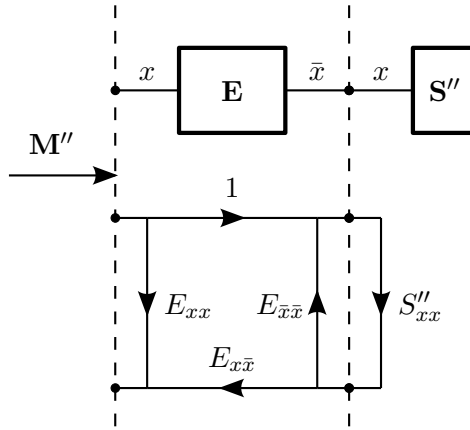


Figure 2: One Port Calibration

The following equation describes the cascading of the reflection error terms and the definition of the calibration standard

$$M''_{xx}{}^{(i)} = E_{xx} + \frac{E_{x\bar{x}} S''_{xx}{}^{(i)}}{1 - E_{\bar{x}\bar{x}} S''_{xx}{}^{(i)}} \quad (12)$$

with

$$\bar{x} = N + x. \quad (13)$$

$N$  is the number of ports and  $x$  is the actual port where the reflection calibration is performed. Equation 12 can be rearranged as

$$\underbrace{M''_{xx}{}^{(i)}}_{y_i} = \underbrace{E_{xx}}_{p_1} + \underbrace{E_{x\bar{x}}}_{p_2} M''_{xx}{}^{(i)} S''_{xx}{}^{(i)} + \underbrace{(E_{x\bar{x}} - E_{xx} E_{\bar{x}\bar{x}})}_{p_3} S''_{xx}{}^{(i)}. \quad (14)$$

Equation 14 can be written as a system of linear equations

$$\mathbf{p} = \mathbf{A}^{-1} \mathbf{y} \quad (15)$$

with

$$\mathbf{A} = \begin{pmatrix} 1 & M''_{xx}{}^{(1)} S''_{xx}{}^{(1)} & S''_{xx}{}^{(1)} \\ 1 & M''_{xx}{}^{(2)} S''_{xx}{}^{(2)} & S''_{xx}{}^{(2)} \\ 1 & M''_{xx}{}^{(3)} S''_{xx}{}^{(3)} & S''_{xx}{}^{(3)} \end{pmatrix} \quad (16)$$



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and

$$\mathbf{y} = \begin{pmatrix} M_{xx}^{(1)} \\ M_{xx}^{(2)} \\ M_{xx}^{(3)} \end{pmatrix}. \quad (17)$$

For example the first measurement could be a short, the second an open and the third a load. The vector  $\mathbf{p}$  contains the solution for the error terms.

$$E_{xx} = p_1 \quad (18)$$

$$E_{\bar{x}x} = 1 \quad (19)$$

$$E_{x\bar{x}} = p_3 + p_1 p_2 \quad (20)$$

$$E_{\bar{x}\bar{x}} = p_2 \quad (21)$$

$E_{xx}$  stands for the directivity,  $E_{x\bar{x}}E_{\bar{x}x}$  denotes the reflection tracking and  $E_{\bar{x}\bar{x}}$  designates the source match term.

### 3.1.2 Power

For a power calibration the scaling factor of the tracking terms have to be determined.

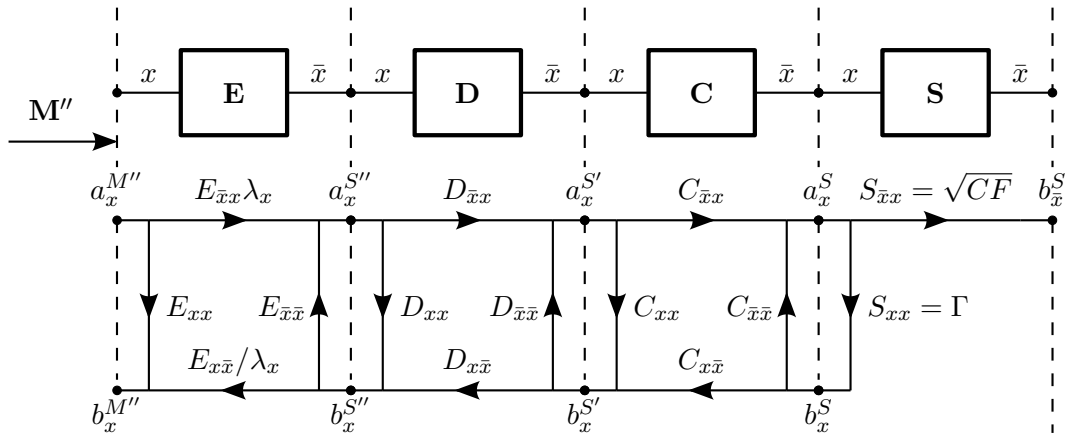


Figure 3: Power Calibration

In a first step the reflection coefficient  $\Gamma$  of the power sensor is computed using error correction, see section 2.

In a second step the scaling factor  $\lambda$  is computed using the receiver values from the VNA, the indicated power of the power sensor, the calibration factor  $CF$  of the power sensor, the error terms and all uncertainty influences. The following set of equations holds

$$b_x^{M''} = a_x^{M''} E_{xx} + b_x^{S''} E_{x\bar{x}} / \lambda \quad (22)$$

$$a_x^{S''} = a_x^{M''} E_{\bar{x}x} \lambda + b_x^{S''} E_{\bar{x}\bar{x}} \quad (23)$$

where

$a_x^{M''}$  denotes the reference receiver value of port  $x$  including noise and linearity uncertainties,

$b_x^{M''}$  denotes the test receiver value of port  $x$  including noise and linearity uncertainties,



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$a_x^{S''}$  denotes the incident wave of the power sensor connected to port  $x$  in  $\sqrt{\text{mW}}$  including drift, cable stability, connector repeatability and standard definition uncertainties,

$b_x^{S''}$  denotes the reflected wave of the power sensor connected to port  $x$  in  $\sqrt{\text{mW}}$  including drift, cable stability, connector repeatability and standard definition uncertainties,

$b_x^S$  denotes the indicated power of the power sensor in mW including all absolute and relative uncertainties of the power sensor,

$E_{xx}$  denotes the directivity of port  $x$ ,

$E_{x\bar{x}}E_{\bar{x}x}$  denotes the reflection tracking of port  $x$ ,

$E_{\bar{x}\bar{x}}$  denotes the source match term of port  $x$ ,

$\lambda$  denotes the scaling factor of the tracking terms,

$D_{xx}, D_{\bar{x}x}, D_{x\bar{x}}, D_{\bar{x}\bar{x}}$  denote the drift influences of the power sensor measurement at port  $x$ ,

$C_{xx}, C_{\bar{x}x}, C_{x\bar{x}}, C_{\bar{x}\bar{x}}$  denote the cable stability and the connector repeatability influences of the power sensor measurement at port  $x$ ,

$S_{xx}$  denotes the reflection coefficient  $\Gamma$  of the power sensor and

$S_{\bar{x}\bar{x}}$  denotes the square root of the calibration factor  $\sqrt{CF}$  of the power sensor, see section 6.18.

One can solve equation 22 for  $b_x^{S''}$

$$b_x^{S''} = \frac{b_x^{M''} - a_x^{M''} E_{xx}}{E_{x\bar{x}}} \lambda \quad (24)$$

and plug the result into 23

$$a_x^{S''} = a_x^{M''} E_{\bar{x}x} \lambda + \frac{b_x^{M''} - a_x^{M''} E_{xx}}{E_{x\bar{x}}} E_{\bar{x}\bar{x}} \lambda. \quad (25)$$

One can solve the above equation for the scaling factor  $\lambda$  of the tracking terms

$$\lambda = \frac{a_x^{S''}}{a_x^{M''} E_{\bar{x}x} + \frac{b_x^{M''} - a_x^{M''} E_{xx}}{E_{x\bar{x}}} E_{\bar{x}\bar{x}}}. \quad (26)$$

The phase of  $\lambda$  is unknown because the phase of the incident wave of the power sensor  $a_x^{S''}$  is unknown. For further data processing the absolute value of  $\lambda$  is used, e.g.: average of multiple power calibrations which finally yields to

$$|\lambda| = \sum_{i=1}^n \left| \frac{a_x^{S''(i)}}{a_x^{M''(i)} E_{\bar{x}x} + \frac{b_x^{M''(i)} - a_x^{M''(i)} E_{xx}}{E_{x\bar{x}}} E_{\bar{x}\bar{x}}} \right| / n. \quad (27)$$



### 3.2 GSOLT Calibration

For a GSOLT calibration the switch terms have to be determined.

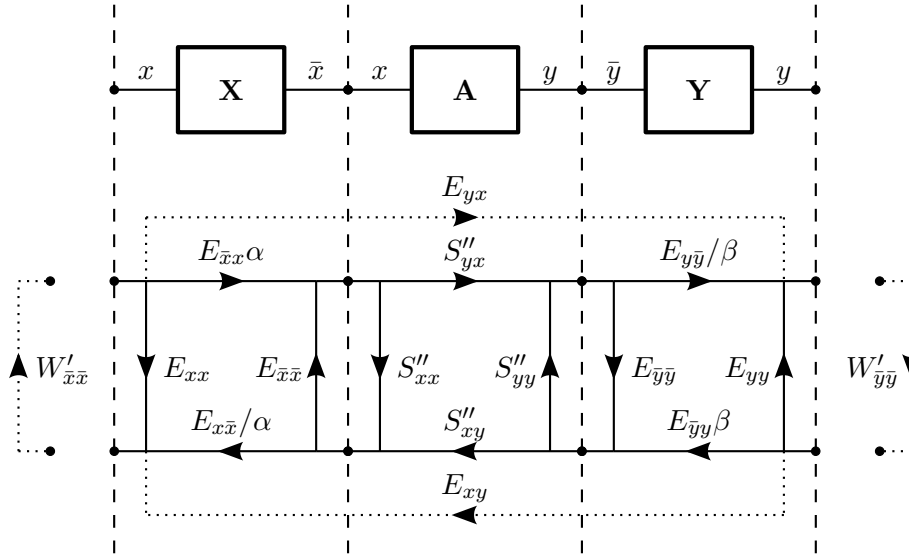


Figure 4: GSOLT Calibration

The following S-parameter matrix describes the transmission standard

$$\mathbf{A}^{(i)} = \begin{pmatrix} S''_{xx}^{(i)} & S''_{xy}^{(i)} \\ S''_{yx}^{(i)} & S''_{yy}^{(i)} \end{pmatrix}. \quad (28)$$

The error box of port  $x$  is denoted as

$$\mathbf{X} = \begin{pmatrix} E_{xx} & E_{x\bar{x}} \\ E_{\bar{x}x} & E_{\bar{x}\bar{x}} \end{pmatrix} \quad (29)$$

with

$$\bar{x} = N + x \quad (30)$$

and the error box of port  $y$  is designated by

$$\mathbf{Y} = \begin{pmatrix} E_{\bar{y}\bar{y}} & E_{\bar{y}y} \\ E_{y\bar{y}} & E_{yy} \end{pmatrix} \quad (31)$$

with

$$\bar{y} = N + y. \quad (32)$$

Cascading the error box of port  $x$ , the transmission standard definition and the error box of port  $y$  yields a new S-parameter matrix.

$$\mathbf{T}^{(i)} = \mathbf{X} \otimes \mathbf{A}^{(i)} \otimes \mathbf{Y} \quad (33)$$

The operator  $\otimes$  denotes the cascading of two 2-ports, as described in appendix A.1.3.

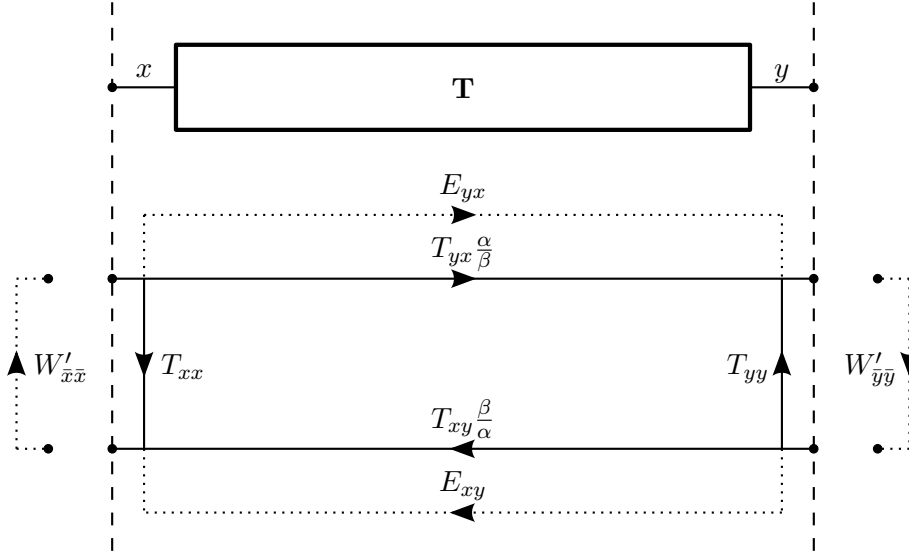


Figure 5: GSOLT Calibration (simplified)

The reflection measurement at port  $x$  is described by

$$M'_{xx}{}^{(i)} = T_{xx}^{(i)} + \frac{\left(\frac{\alpha^{(i)}}{\beta^{(i)}} T_{yx}^{(i)} + E_{yx}\right) \left(\frac{\beta^{(i)}}{\alpha^{(i)}} T_{xy}^{(i)} + E_{xy}\right) W'_{yy}{}^{(i)}}{1 - T_{yy}^{(i)} W'_{yy}{}^{(i)}}, \quad (34)$$

the transmission measurement from port  $x$  to  $y$  is described by

$$M'_{yx}{}^{(i)} = \frac{\left(\frac{\alpha^{(i)}}{\beta^{(i)}} T_{yx}^{(i)} + E_{yx}\right)}{1 - T_{yy}^{(i)} W'_{yy}{}^{(i)}}, \quad (35)$$

the transmission measurement from port  $y$  to  $x$  is described by

$$M'_{xy}{}^{(i)} = \frac{\left(\frac{\beta^{(i)}}{\alpha^{(i)}} T_{xy}^{(i)} + E_{xy}\right)}{1 - T_{xx}^{(i)} W'_{xx}{}^{(i)}}, \quad (36)$$

and the reflection measurement at port  $y$  is described by

$$M'_{yy}{}^{(i)} = T_{yy}^{(i)} + \frac{\left(\frac{\alpha^{(i)}}{\beta^{(i)}} T_{yx}^{(i)} + E_{yx}\right) \left(\frac{\beta^{(i)}}{\alpha^{(i)}} T_{xy}^{(i)} + E_{xy}\right) W'_{xx}{}^{(i)}}{1 - T_{xx}^{(i)} W'_{xx}{}^{(i)}}. \quad (37)$$

### 3.2.1 Switch Terms without Crosstalk

If the crosstalk  $E_{yx}$  and  $E_{xy}$  are assumed as zero, equations 34 and 37 can be simplified as

$$M'_{xx}{}^{(thru)} = T_{xx}{}^{(thru)} + \frac{T_{yx}{}^{(thru)} T_{xy}{}^{(thru)} W'_{yy}{}^{(thru)}}{1 - T_{yy}{}^{(thru)} W'_{yy}{}^{(thru)}} \quad (38)$$



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and

$$M'_{yy}{}^{(thru)} = T_{yy}{}^{(thru)} + \frac{T_{yx}{}^{(thru)}T_{xy}{}^{(thru)}W'_{\bar{x}\bar{x}}{}^{(thru)}}{1 - T_{xx}{}^{(thru)}W'_{\bar{x}\bar{x}}{}^{(thru)}}. \quad (39)$$

Equation 39 can be rearranged as

$$\left(M'_{yy}{}^{(thru)} - T_{yy}{}^{(thru)}\right) \left(1 - T_{xx}{}^{(thru)}W'_{\bar{x}\bar{x}}{}^{(thru)}\right) = T_{yx}{}^{(thru)}T_{xy}{}^{(thru)}W'_{\bar{x}\bar{x}}{}^{(thru)} \quad (40)$$

and one can solve for the switch term of port  $x$

$$W'_{\bar{x}\bar{x}}{}^{(thru)} = \frac{M'_{yy}{}^{(thru)} - T_{yy}{}^{(thru)}}{\left(M'_{yy}{}^{(thru)} - T_{yy}{}^{(thru)}\right)T_{xx}{}^{(thru)} + T_{yx}{}^{(thru)}T_{xy}{}^{(thru)}}. \quad (41)$$

The other switch term of port  $y$  can be calculated with the following equation

$$W'_{\bar{y}\bar{y}}{}^{(thru)} = \frac{M'_{xx}{}^{(thru)} - T_{xx}{}^{(thru)}}{\left(M'_{xx}{}^{(thru)} - T_{xx}{}^{(thru)}\right)T_{yy}{}^{(thru)} + T_{yx}{}^{(thru)}T_{xy}{}^{(thru)}}. \quad (42)$$

Finally one can remove the drift effects from the switch terms.

$$W_{\bar{x}\bar{x}} = W'_{\bar{x}\bar{x}}{}^{(thru)} - V_{\bar{x}\bar{x}}{}^{(thru)} \quad (43)$$

$$W_{\bar{y}\bar{y}} = W'_{\bar{y}\bar{y}}{}^{(thru)} - V_{\bar{y}\bar{y}}{}^{(thru)} \quad (44)$$

### 3.2.2 Switch Terms with Crosstalk

A new auxiliary variable  $m_x$  is introduced by combining equations 34, 35 and 36

$$m_x = \frac{M'_{xx}{}^{(thru)} - T_{xx}{}^{(thru)}}{M'_{yx}{}^{(thru)}M'_{xy}{}^{(thru)}} = \left(1 - T_{xx}{}^{(thru)}W'_{\bar{x}\bar{x}}{}^{(thru)}\right)W'_{\bar{y}\bar{y}}{}^{(thru)} \quad (45)$$

and  $m_y$  by combining equations 35, 36 and 37

$$m_y = \frac{M'_{yy}{}^{(thru)} - T_{yy}{}^{(thru)}}{M'_{yx}{}^{(thru)}M'_{xy}{}^{(thru)}} = \left(1 - T_{yy}{}^{(thru)}W'_{\bar{y}\bar{y}}{}^{(thru)}\right)W'_{\bar{x}\bar{x}}{}^{(thru)}. \quad (46)$$

Now one has a system with two equations and two unknown variables, which are both switch terms. Equation 45 can be rewritten as

$$W'_{\bar{y}\bar{y}}{}^{(thru)} = \frac{m_x}{1 - T_{xx}{}^{(thru)}W'_{\bar{x}\bar{x}}{}^{(thru)}} \quad (47)$$

and one can put equation 47 into equation 46

$$m_y = \left(1 - T_{yy}{}^{(thru)}\frac{m_x}{1 - T_{xx}{}^{(thru)}W'_{\bar{x}\bar{x}}{}^{(thru)}}\right)W'_{\bar{x}\bar{x}}{}^{(thru)}. \quad (48)$$

One can find the root of equation 48

$$\underbrace{T_{xx}{}^{(thru)}}_a \left(W'_{\bar{x}\bar{x}}{}^{(thru)}\right)^2 + \underbrace{\left(m_x T_{yy}{}^{(thru)} - m_y T_{xx}{}^{(thru)} - 1\right)}_b W'_{\bar{x}\bar{x}}{}^{(thru)} + \underbrace{m_y}_c = 0 \quad (49)$$



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and solving the quadratic equation yields the switch term of port  $x$ . The other switch term of port  $y$  can be calculated with equation 47.

$$W_{\bar{x}\bar{x}}^{(thru)} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (50)$$

Finally one can remove the drift effects from the switch terms.

$$W_{\bar{x}\bar{x}} = W_{\bar{x}\bar{x}}^{(thru)} - V_{\bar{x}\bar{x}}^{(thru)} \quad (51)$$

$$W_{\bar{y}\bar{y}} = W_{\bar{y}\bar{y}}^{(thru)} - V_{\bar{y}\bar{y}}^{(thru)} \quad (52)$$

### 3.2.3 Crosstalk

If the switch terms are known, the crosstalk between two ports can be measured directly. Note that it is assumed that there is no drift of the isolation.

$$E_{yx} = M_{yx}^{(isol)} \quad (53)$$

$$E_{xy} = M_{xy}^{(isol)} \quad (54)$$

### 3.2.4 Symmetry

The following equation defines  $\alpha$ .

$$\alpha = 1 \quad (55)$$

To find a solution for  $\beta$ , equation 35 can be rewritten as

$$\beta_1 = \alpha \frac{T_{yx}^{(thru)}}{M_{yx}^{(thru)} \left( 1 - T_{yy}^{(thru)} W_{\bar{y}\bar{y}}^{(thru)} \right) - E_{yx}} \quad (56)$$

and equation 36 can be rewritten as

$$\beta_2 = \alpha \frac{M_{xy}^{(thru)} \left( 1 - T_{xx}^{(thru)} W_{\bar{x}\bar{x}}^{(thru)} \right) - E_{xy}}{T_{xy}^{(thru)}}. \quad (57)$$

The GSOLT calibration algorithm assumes  $\beta$  as the mean of the forward  $\beta_1$  and the reverse  $\beta_2$

$$\beta = \frac{\beta_1 + \beta_2}{2} \quad (58)$$

then one can update the reflection and transmission tracking error terms of port  $x$  and  $y$ .

$$E_{x\bar{x}} := \frac{E_{x\bar{x}}}{\alpha} \quad (59)$$

$$E_{\bar{x}x} := E_{\bar{x}x} \alpha \quad (60)$$

$$E_{y\bar{y}} := \frac{E_{y\bar{y}}}{\beta} \quad (61)$$

$$E_{\bar{y}y} := E_{\bar{y}y} \beta \quad (62)$$

$E_{x\bar{x}} E_{\bar{x}x}$  stands for the reflection tracking of port  $x$ ,

$E_{y\bar{y}} E_{\bar{y}y}$  stands for the reflection tracking of port  $y$ ,

$E_{x\bar{x}} E_{\bar{y}y}$  stands for the transmission tracking from port  $y$  to port  $x$  and

$E_{y\bar{y}} E_{\bar{x}x}$  stands for the transmission tracking from port  $x$  to port  $y$ .



### 3.3 QSOLT Calibration

The QSOLT calibration is described in [11].

#### 3.3.1 Crosstalk

The crosstalk between two ports can be measured directly. Note that it is assumed that there is no drift of the isolation.

$$E_{yx} = M_{yx}^{(isol)} \quad (63)$$

$$E_{xy} = M_{xy}^{(isol)} \quad (64)$$

#### 3.3.2 Copy Calibration

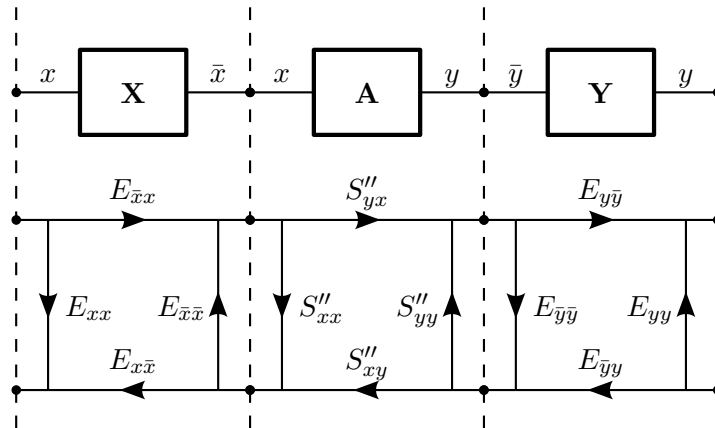


Figure 6: QSOLT Calibration

The following S-parameter matrix describes the switch term and crosstalk corrected measured data of the transmission standard

$$\mathbf{T}^{(i)} = \begin{pmatrix} M_{xx}^{(i)} & M_{xy}^{(i)} - E_{xy} \\ M_{yx}^{(i)} - E_{yx} & M_{yy}^{(i)} \end{pmatrix}. \quad (65)$$

Again the transmission standard is defined as

$$\mathbf{A}^{(i)} = \begin{pmatrix} S_{xx}^{(i)} & S_{xy}^{(i)} \\ S_{yx}^{(i)} & S_{yy}^{(i)} \end{pmatrix}. \quad (66)$$

The error box of port  $x$  is defined as

$$\mathbf{X} = \begin{pmatrix} E_{xx} & E_{x\bar{x}} \\ E_{\bar{x}x} & E_{\bar{x}\bar{x}} \end{pmatrix} \quad (67)$$

with

$$\bar{x} = N + x \quad (68)$$



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and the error box of port  $y$  is defined as

$$\mathbf{Y} = \begin{pmatrix} E_{\bar{y}\bar{y}} & E_{\bar{y}y} \\ E_{y\bar{y}} & E_{yy} \end{pmatrix} \quad (69)$$

with

$$\bar{y} = N + y. \quad (70)$$

To copy the error terms of port  $x$  to port  $y$  one decascades  $\mathbf{X}$  and  $\mathbf{A}$  from  $\mathbf{T}$ .

$$\mathbf{Y} = \left( \mathbf{X} \otimes \mathbf{A}^{(thru)} \right)^{\ominus 1} \otimes \mathbf{T}^{(thru)} \quad (71)$$

The operator  $\otimes$  denotes the cascading of two 2-ports, as described in appendix A.1.3.

### 3.4 Unknown Thru Calibration

The Unknown Thru calibration is described in [12].

#### 3.4.1 Crosstalk

The crosstalk between two ports can be measured directly. Note that it is assumed that there is no drift of the isolation.

$$E_{yx} = M_{yx}^{(isol)} \quad (72)$$

$$E_{xy} = M_{xy}^{(isol)} \quad (73)$$

#### 3.4.2 Symmetry

The following S-parameter matrix describes the error box of port  $x$  and  $y$ .

$$\mathbf{E}' = \begin{pmatrix} E_{xx} & E_{xy} & E_{x\bar{x}} & E_{x\bar{y}} \\ E_{yx} & E_{yy} & E_{y\bar{x}} & E_{y\bar{y}} \\ E_{\bar{x}x} & E_{\bar{x}y} & E_{\bar{x}\bar{x}} & E_{\bar{x}\bar{y}} \\ E_{\bar{y}x} & E_{\bar{y}y} & E_{\bar{y}\bar{x}} & E_{\bar{y}\bar{y}} \end{pmatrix} \quad (74)$$

One can error correct the unknown thru measurement data without knowing the symmetry error terms.

$$\mathbf{S}^{(thru)} = \mathbf{M}^{(thru)} \ominus \mathbf{E}' \quad (75)$$

The operator  $\ominus$  denotes the decascading of two S-parameter sets, as described in appendix A.1.1.

The forward and reverse transmission S-parameter of an unknown thru have to be the same, because the unknown thru is assumed to be a reciprocal device. The nominal magnitude of the transmission of the unknown thru is described by

$$|s_{yx}| = \sqrt{|S_{yx}^{(thru)} S_{xy}^{(thru)}|} \quad (76)$$



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and the nominal phase is described by

$$\arg(s_{yx}) = \frac{\arg\left(S_{yx}''(thru) S_{xy}''(thru)\right)}{2}. \quad (77)$$

The following equation defines  $\alpha$ .

$$\alpha = 1 \quad (78)$$

One can compute  $\beta$  with the nominal transmission and the error corrected unknown thru measurement.

$$\beta = \alpha \frac{s_{yx}}{S_{yx}''(thru)} \quad (79)$$

Then one can update the reflection and transmission tracking error terms of port  $x$  and  $y$ .

$$E_{x\bar{x}} := \frac{E_{x\bar{x}}}{\alpha} \quad (80)$$

$$E_{\bar{x}x} := E_{\bar{x}x} \alpha \quad (81)$$

$$E_{y\bar{y}} := \frac{E_{y\bar{y}}}{\beta} \quad (82)$$

$$E_{\bar{y}y} := E_{\bar{y}y} \beta \quad (83)$$

$E_{x\bar{x}} E_{\bar{x}x}$  is the reflection tracking of port  $x$ ,

$E_{y\bar{y}} E_{\bar{y}y}$  is the reflection tracking of port  $y$ ,

$E_{x\bar{x}} E_{\bar{y}y}$  is the transmission tracking from port  $y$  to port  $x$  and

$E_{y\bar{y}} E_{\bar{x}x}$  is the transmission tracking from port  $x$  to port  $y$ .

### 3.5 TRL Calibration

The TRL calibration is described in [13] works only with two ports.

#### 3.5.1 Crosstalk

The crosstalk between two ports can be measured directly. Note that it is assumed that there is no drift of the isolation.

$$E_{yx} = M_{yx}''(isol) \quad (84)$$

$$E_{xy} = M_{xy}''(isol) \quad (85)$$

#### 3.5.2 Thru Reflect Line

The following T-parameter matrix describes the measurement of the thru which is corrected for switch terms and crosstalk.

$$\mathbf{T}_{thru} = \text{StoTParam} \left( \begin{array}{cc} M_{xx}''(thru) & M_{xy}''(thru) - E_{xy} \\ M_{yx}''(thru) - E_{yx} & M_{yy}''(thru) \end{array} \right) \quad (86)$$



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The next matrix describes the measurement of the line which is corrected for switch terms and crosstalk.

$$\mathbf{T}_{line} = \text{StoTParam} \begin{pmatrix} M_{xx}''(line) & M_{xy}''(line) - E_{xy} \\ M_{yx}''(line) - E_{yx} & M_{yy}''(line) \end{pmatrix} \quad (87)$$

One can cascade the line and the inverted thru.

$$\mathbf{m} = \mathbf{T}_{line} \times \mathbf{T}_{thru}^{-1} \quad (88)$$

With the elements of the matrix  $\mathbf{m}$  one forms

$$a_m = m_{21} \quad (89)$$

$$b_m = m_{22} - m_{11} \quad (90)$$

$$c_m = -m_{12}. \quad (91)$$

$$x_1 = \frac{-b_m + \sqrt{b_m^2 - 4a_m c_m}}{2a_m} \quad (92)$$

$$x_2 = \frac{-b_m - \sqrt{b_m^2 - 4a_m c_m}}{2a_m} \quad (93)$$

If  $|x_1| > |x_2|$  then one sets

$$\alpha = x_1 \quad (94)$$

$$\beta = x_2 \quad (95)$$

else one makes the inverse assignment

$$\alpha = x_2 \quad (96)$$

$$\beta = x_1. \quad (97)$$

One can cascade the inverted thru and the line.

$$\mathbf{n} = \mathbf{T}_{thru}^{-1} \times \mathbf{T}_{line} \quad (98)$$

With the elements of the matrix  $\mathbf{n}$  one forms

$$a_n = n_{12} \quad (99)$$

$$b_n = n_{22} - n_{11} \quad (100)$$

$$c_n = -n_{21}. \quad (101)$$

$$y_1 = \frac{-b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n} \quad (102)$$

$$y_2 = \frac{-b_n - \sqrt{b_n^2 - 4a_n c_n}}{2a_n} \quad (103)$$



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If  $|y_1| > |y_2|$  then one sets

$$\gamma = y_1 \quad (104)$$

$$\delta = y_2 \quad (105)$$

else one makes the inverse assignment

$$\gamma = y_2 \quad (106)$$

$$\delta = y_1. \quad (107)$$

The TRL algorithm additionally uses the following intermediate quantities.

$$f_1 = \frac{\beta - M_{xx}^{(reflect)}}{\alpha - M_{xx}^{(reflect)}} \quad (108)$$

$$f_2 = \frac{\gamma + M_{yy}^{(reflect)}}{\delta + M_{yy}^{(reflect)}} \quad (109)$$

$$f_3 = \frac{\beta - M_{xx}^{(thru)}}{\alpha - M_{xx}^{(thru)}} \quad (110)$$

Now one can determine the parameters of the cascaded error boxes. The sign of  $e_{11}$  is chosen by the approximate definition of the reflection standard.

$$e_{00} = \beta \quad (111)$$

$$e_{33} = -\delta \quad (112)$$

$$e_{11} = \pm \sqrt{f_1 f_2 f_3} \quad (113)$$

$$e_{22} = \frac{f_3}{e_{11}} \quad (114)$$

$$e_{1001} = (\beta - a) e_{11} \quad (115)$$

$$e_{2332} = (\gamma - d) e_{22} \quad (116)$$

$$e_{1032} = \left( M_{yx}^{(thru)} - E_{yx} \right) (1 - e_{11} e_{22}) \quad (117)$$

$$e_{2301} = \left( M_{xy}^{(thru)} - E_{xy} \right) (1 - e_{11} e_{22}) \quad (118)$$

Now the offset delay  $d = e^{-\gamma l}$  of the thru definition is removed from the error boxes.

$$e_{11} := e_{11}/d \quad (119)$$

$$e_{22} := e_{22}/d \quad (120)$$

$$e_{1001} := e_{1001}/d \quad (121)$$

$$e_{2332} := e_{2332}/d \quad (122)$$

$$e_{1032} := e_{1032}/d \quad (123)$$

$$e_{2301} := e_{2301}/d \quad (124)$$



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Setting one transmission term to 1 defines the remaining terms.

$$e_{10} = 1 \quad (125)$$

$$e_{01} = \frac{e_{1001}}{e_{10}} \quad (126)$$

$$e_{32} = \frac{e_{1032}}{e_{10}} \quad (127)$$

$$e_{23} = \frac{e_{2301}}{e_{01}} \quad (128)$$

Finally one can update the error terms of the VNA measurement model.

$$E_{xx} = e_{00} \quad (129)$$

$$E_{\bar{x}x} = e_{10} \quad (130)$$

$$E_{x\bar{x}} = e_{01} \quad (131)$$

$$E_{\bar{x}\bar{x}} = e_{11} \quad (132)$$

$$E_{\bar{y}\bar{y}} = e_{22} \quad (133)$$

$$E_{y\bar{y}} = e_{32} \quad (134)$$

$$E_{\bar{y}y} = e_{23} \quad (135)$$

$$E_{yy} = e_{33} \quad (136)$$

$E_{xx}$  is the directivity of port  $x$ ,

$E_{\bar{x}\bar{x}}$  is the match of port  $x$ ,

$E_{x\bar{x}}E_{\bar{x}x}$  is the reflection tracking of port  $x$ ,

$E_{y\bar{y}}E_{\bar{x}x}$  is the transmission tracking from port  $x$  to port  $y$ ,

$E_{x\bar{x}}E_{\bar{y}y}$  is the transmission tracking from port  $y$  to port  $x$ ,

$E_{y\bar{y}}E_{\bar{y}y}$  is the reflection tracking of port  $y$ ,

$E_{\bar{y}\bar{y}}$  is the match of port  $y$  and

$E_{yy}$  is the directivity of port  $y$ .



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### 3.6 LRRM Calibration

The in METAS VNA Tools implemented LRRM algorithm is a generalization of the LRRM calibration described in [14]. The generalization consists of implementing a reflective non-reciprocal line standard. The line standard has to be fully known. The two reflection standards are measured each at both ports and have to have different reflection angles or amplitudes. The match standard is only measured at one port. The DC resistance  $R$  of the match has to be known. The following table describes the used calibration standards in the LRRM calibration.

Table 1: LRRM Standards

Standard	S-parameter
Line	$\mathbf{S}^{(t)} = \begin{pmatrix} s_{xx}^{(t)} & s_{xy}^{(t)} \\ s_{yx}^{(t)} & s_{yy}^{(t)} \end{pmatrix}$
Reflect 1 (unknown capacitance)	$S^{(o)} = \frac{1-j\omega CZ_r}{1+j\omega CZ_r}$ with $C = ?$ and $C \geq 0$
Reflect 2 (unknown reflection)	$S^{(r)} = r_{re} + jr_{im}$ with $r_{re} = ?$ , $r_{im} = ?$ and $r_{re} \leq 0$
Match (unknown inductance)	$S^{(m)} = \frac{R+j\omega L-Z_r}{R+j\omega L+Z_r}$ with $L = ?$

The LRRM calibration can be described by the following nine linear equations, see section 3.8.

$$\begin{pmatrix} M_{xx}^{(t)} & 0 & 1 & 0 & M_{xx}^{(t)} S_{xx}^{(t)} & M_{xy}^{(t)} S_{yx}^{(t)} & S_{xx}^{(t)} & 0 \\ M_{yx}^{(t)} & 0 & 0 & 0 & M_{yx}^{(t)} S_{xx}^{(t)} & M_{yy}^{(t)} S_{yx}^{(t)} & 0 & S_{yx}^{(t)} \\ 0 & M_{xy}^{(t)} & 0 & 0 & M_{xx}^{(t)} S_{xy}^{(t)} & M_{xy}^{(t)} S_{yy}^{(t)} & S_{xy}^{(t)} & 0 \\ 0 & M_{yy}^{(t)} & 0 & 1 & M_{yx}^{(t)} S_{xy}^{(t)} & M_{yy}^{(t)} S_{yy}^{(t)} & 0 & S_{yy}^{(t)} \\ \hline M_{xx}^{(o)} & 0 & 1 & 0 & M_{xx}^{(o)} S_{xx}^{(o)} & 0 & S_{xx}^{(o)} & 0 \\ 0 & M_{yy}^{(o)} & 0 & 1 & 0 & M_{yy}^{(o)} S_{yy}^{(o)} & 0 & S_{yy}^{(o)} \\ \hline M_{xx}^{(r)} & 0 & 1 & 0 & M_{xx}^{(r)} S_{xx}^{(r)} & 0 & S_{xx}^{(r)} & 0 \\ 0 & M_{yy}^{(r)} & 0 & 1 & 0 & M_{yy}^{(r)} S_{yy}^{(r)} & 0 & S_{yy}^{(r)} \\ \hline M_{xx}^{(m)} & 0 & 1 & 0 & M_{xx}^{(m)} S_{xx}^{(m)} & 0 & S_{xx}^{(m)} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = 0 \quad (137)$$

The LRRM calibration is solved with the following steps:

1. The seven unknowns  $x_2$  to  $x_7$  which represent the error terms can be eliminated. This yields two linear equations.



2. The numerators of these two equations have to be equal to zero:

$$\begin{aligned}
 & M_{xx}''(o) ((M_{yy}''(o) - M_{yy}''(t)) (M_{xx}''(r) (S_{xx}''(o) - S_{xx}''(r)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(m) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) \\
 & - M_{xx}''(t) (S_{xx}''(m) - S_{xx}''(r)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(o) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) + M_{xx}''(m) (S_{xx}''(o) \\
 & - S_{xx}''(o)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(r) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) - M_{xy}''(t) M_{yx}''(t) (S_{xx}''(m) \\
 & - S_{xx}''(r)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(o) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) + M_{xx}''(r) (- (M_{yy}''(o) \\
 & - M_{yy}''(t)) (M_{xx}''(m) (S_{xx}''(m) - S_{xx}''(r)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(o) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) \\
 & - M_{xx}''(t) (S_{xx}''(m) - S_{xx}''(o)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(r) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) \\
 & + M_{xy}''(t) M_{yx}''(t) (S_{xx}''(m) - S_{xx}''(o)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(r) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) \\
 & + M_{xx}''(m) (M_{xy}''(t) M_{yx}''(t) + M_{xx}''(t) (M_{yy}''(o) - M_{yy}''(t))) (S_{xx}''(o) - S_{xx}''(r)) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(m) \\
 & - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) = 0
 \end{aligned} \tag{138}$$

and

$$\begin{aligned}
 & (M_{xy}''(t) M_{yx}''(t) + M_{xx}''(t) (M_{yy}''(o) - M_{yy}''(t))) (M_{xy}''(t) M_{yx}''(t) + M_{xx}''(t) (M_{yy}''(r) - M_{yy}''(t))) (S_{xx}''(m) \\
 & - S_{xx}''(o)) S_{xy}''(t) S_{yx}''(t) (S_{yy}''(o) - S_{yy}''(r)) + M_{xx}''(o) (M_{xx}''(t) (M_{yy}''(o) - M_{yy}''(t)) (-M_{yy}''(r) \\
 & + M_{yy}''(t)) (S_{xx}''(m) - S_{xx}''(o)) S_{xy}''(t) S_{yx}''(t) (S_{yy}''(o) - S_{yy}''(r)) + M_{xy}''(t) M_{yx}''(t) (M_{yy}''(t) (S_{xx}''(m) \\
 & - S_{xx}''(o)) S_{xy}''(t) S_{yx}''(t) (S_{yy}''(o) - S_{yy}''(r)) - M_{yy}''(r) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(o) - S_{xx}''(t)) (S_{yy}''(o) \\
 & - S_{yy}''(t))) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(m) - S_{xx}''(t)) (S_{yy}''(r) - S_{yy}''(t))) + M_{yy}''(o) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(m) \\
 & - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(o) - S_{xx}''(t)) (S_{yy}''(r) - S_{yy}''(t))) + M_{xx}''(m) ((M_{xx}''(o) \\
 & - M_{xx}''(t)) (M_{yy}''(o) - M_{yy}''(t)) (M_{yy}''(r) - M_{yy}''(t)) (S_{xx}''(m) - S_{xx}''(o)) S_{xy}''(t) S_{yx}''(t) (S_{yy}''(o) - S_{yy}''(r)) \\
 & + M_{xy}''(t) M_{yx}''(t) (M_{yy}''(t) (S_{xx}''(m) - S_{xx}''(o)) S_{xy}''(t) S_{yx}''(t) (S_{yy}''(o) - S_{yy}''(r)) - M_{yy}''(o) (S_{xy}''(t) S_{yx}''(t) \\
 & - (S_{xx}''(o) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(m) - S_{xx}''(t)) (S_{yy}''(r) - S_{yy}''(t))) \\
 & + M_{yy}''(r) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(m) - S_{xx}''(t)) (S_{yy}''(o) - S_{yy}''(t))) (S_{xy}''(t) S_{yx}''(t) - (S_{xx}''(o) - S_{xx}''(t)) (S_{yy}''(r) \\
 & - S_{yy}''(t)))) = 0.
 \end{aligned} \tag{139}$$

3.  $S_{xx}''(o)$  and  $S_{yy}''(o)$  can be replaced by the unknown capacitance  $C$ :

$$S_{xx}''(o) = D_{xx}^{(o)} + \frac{D_{\bar{x}x}^{(o)} S_{xx}'^{(o)} D_{x\bar{x}}^{(o)}}{1 - D_{\bar{x}\bar{x}}^{(o)} S_{xx}'^{(o)}} \text{ with } S_{xx}'^{(o)} = C_{xx}^{(o)} + \frac{C_{\bar{x}x}^{(o)} S^{(o)} C_{x\bar{x}}^{(o)}}{1 - C_{\bar{x}\bar{x}}^{(o)} S^{(o)}} \tag{140}$$

and

$$S_{yy}''(o) = D_{yy}^{(o)} + \frac{D_{\bar{y}y}^{(o)} S_{yy}'^{(o)} D_{y\bar{y}}^{(o)}}{1 - D_{\bar{y}\bar{y}}^{(o)} S_{yy}'^{(o)}} \text{ with } S_{yy}'^{(o)} = C_{yy}^{(o)} + \frac{C_{\bar{y}y}^{(o)} S^{(o)} C_{y\bar{y}}^{(o)}}{1 - C_{\bar{y}\bar{y}}^{(o)} S^{(o)}} \tag{141}$$

where

$$S^{(o)} = \frac{1 - j\omega CZ_r}{1 + j\omega CZ_r}. \tag{142}$$

Here  $C_{xx}^{(o)}$ ,  $C_{\bar{x}x}^{(o)}$ ,  $C_{x\bar{x}}^{(o)}$  and  $C_{\bar{x}\bar{x}}^{(o)}$  are cable influences and  $D_{xx}^{(o)}$ ,  $D_{\bar{x}x}^{(o)}$ ,  $D_{x\bar{x}}^{(o)}$  and  $D_{\bar{x}\bar{x}}^{(o)}$  are drift influences of the reflection standard 1 (open) measurement at port  $x$ .

$C_{yy}^{(o)}$ ,  $C_{\bar{y}y}^{(o)}$ ,  $C_{y\bar{y}}^{(o)}$  and  $C_{\bar{y}\bar{y}}^{(o)}$  are cable influences and  $D_{yy}^{(o)}$ ,  $D_{\bar{y}y}^{(o)}$ ,  $D_{y\bar{y}}^{(o)}$  and  $D_{\bar{y}\bar{y}}^{(o)}$  are drift influences of the reflection standard 1 (open) measurement at port  $y$ .



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4.  $S''_{xx}^{(r)}$  and  $S''_{yy}^{(r)}$  can be replaced by the unknown reflection  $r_{re}$  and  $r_{im}$ :

$$S''_{xx}^{(r)} = D_{xx}^{(r)} + \frac{D_{\bar{x}x}^{(r)} S'_{xx}^{(r)} D_{x\bar{x}}^{(r)}}{1 - D_{\bar{x}\bar{x}}^{(r)} S'_{xx}^{(r)}} \text{ with } S'_{xx}^{(r)} = C_{xx}^{(r)} + \frac{C_{\bar{x}x}^{(r)} S^{(r)} C_{x\bar{x}}^{(r)}}{1 - C_{\bar{x}\bar{x}}^{(r)} S^{(r)}} \quad (143)$$

and

$$S''_{yy}^{(r)} = D_{yy}^{(r)} + \frac{D_{\bar{y}y}^{(r)} S'_{yy}^{(r)} D_{y\bar{y}}^{(r)}}{1 - D_{\bar{y}\bar{y}}^{(r)} S'_{yy}^{(r)}} \text{ with } S'_{yy}^{(r)} = C_{yy}^{(r)} + \frac{C_{\bar{y}y}^{(r)} S^{(r)} C_{y\bar{y}}^{(r)}}{1 - C_{\bar{y}\bar{y}}^{(r)} S^{(r)}} \quad (144)$$

where

$$S^{(r)} = r_{re} + jr_{im}. \quad (145)$$

Here  $C_{xx}^{(r)}$ ,  $C_{\bar{x}x}^{(r)}$ ,  $C_{x\bar{x}}^{(r)}$  and  $C_{\bar{x}\bar{x}}^{(r)}$  are cable influences and  $D_{xx}^{(r)}$ ,  $D_{\bar{x}x}^{(r)}$ ,  $D_{x\bar{x}}^{(r)}$  and  $D_{\bar{x}\bar{x}}^{(r)}$  are drift influences of the reflection standard 2 measurement at port  $x$ .

$C_{yy}^{(r)}$ ,  $C_{\bar{y}y}^{(r)}$ ,  $C_{y\bar{y}}^{(r)}$  and  $C_{\bar{y}\bar{y}}^{(r)}$  are cable influences and  $D_{yy}^{(r)}$ ,  $D_{\bar{y}y}^{(r)}$ ,  $D_{y\bar{y}}^{(r)}$  and  $D_{\bar{y}\bar{y}}^{(r)}$  are drift influences of the reflection standard 2 measurement at port  $y$ .

5.  $S''_{xx}^{(m)}$  can be replaced by the unknown series inductance  $L$ :

$$S''_{xx}^{(m)} = D_{xx}^{(m)} + \frac{D_{\bar{x}x}^{(m)} S'_{xx}^{(m)} D_{x\bar{x}}^{(m)}}{1 - D_{\bar{x}\bar{x}}^{(m)} S'_{xx}^{(m)}} \text{ with } S'_{xx}^{(m)} = C_{xx}^{(m)} + \frac{C_{\bar{x}x}^{(m)} S^{(m)} C_{x\bar{x}}^{(m)}}{1 - C_{\bar{x}\bar{x}}^{(m)} S^{(m)}} \quad (146)$$

and

$$S^{(m)} = \frac{R + j\omega L - Z_r}{R + j\omega L + Z_r}. \quad (147)$$

Here  $C_{xx}^{(m)}$ ,  $C_{\bar{x}x}^{(m)}$ ,  $C_{x\bar{x}}^{(m)}$  and  $C_{\bar{x}\bar{x}}^{(m)}$  are cable influences and  $D_{xx}^{(m)}$ ,  $D_{\bar{x}x}^{(m)}$ ,  $D_{x\bar{x}}^{(m)}$  and  $D_{\bar{x}\bar{x}}^{(m)}$  are drift influences of the match measurement at port  $x$ .

6. This finally yields two complex non-linear equations with four scalar unknowns.
7. Non-linear optimization yields  $C$ ,  $r_{re}$ ,  $r_{im}$  and  $L$ . This optimization is independent of the error terms. The starting values are  $C = 0$ ,  $r_{re} = -1$ ,  $r_{im} = 0$  and  $L = 0$ .
8. Now all standards are fully known and the QSOLT calibration, see section 3.3, is used to compute the error terms.

### 3.7 Juroshek Calibration

The Juroshek calibration is described in [15]. The following assignment prepares the raw data measured by the VNA.

$$M_{xx}^{(i)} := \frac{M_{xx}^{(i)}}{M_{yx}^{(i)}} \quad (148)$$

Finally one can compute a one port calibration, see section 3.1. Where

$x$  is the VNA port where port 1 of the splitter is connected,

$y$  is the VNA port where port 2 or 3 of the splitter is connected,

$E_{\bar{x}\bar{x}}$  is the equivalent source match of port 3 or 2 of the splitter.



### 3.8 LHKM Calibration

The LHKM calibration is described in [16], [17]. This calibration is not fully implemented in METAS VNA Tools and it's still under development. At the current state it only supports the non-leaky error model and not the full-leaky error model.

The following equation describes the in METAS VNA Tools used LHKM calibration model

$$\mathbf{M}''^{(i)} \mathbf{A} - \mathbf{B} - \mathbf{M}''^{(i)} \mathbf{C} \mathbf{S}''^{(i)} + \mathbf{D} \mathbf{S}''^{(i)} = 0 \quad (149)$$

with

$$\mathbf{A}_{00} = 1 \quad (150)$$

where  $\mathbf{M}''$  denotes the switch corrected measured data including noise and linearity influences. And  $\mathbf{S}''$  denotes the actual data including the cable stability, connector repeatability and the drift of the calibration error terms.

The ABCD terms of the LHKM calibration model can be converted to the generic calibration model. The result will be the error terms  $\mathbf{E}$

$$\mathbf{E} = \begin{pmatrix} \mathbf{E}_{00} & \mathbf{E}_{01} \\ \mathbf{E}_{10} & \mathbf{E}_{11} \end{pmatrix} \quad (151)$$

with

$$\mathbf{E}_{10} = \mathbf{A}^{-1} \quad (152)$$

$$\mathbf{E}_{00} = \mathbf{B} \mathbf{A}^{-1} \quad (153)$$

$$\mathbf{E}_{11} = \mathbf{A}^{-1} \mathbf{C} \quad (154)$$

$$\mathbf{E}_{01} = \mathbf{B} \mathbf{A}^{-1} \mathbf{C} - \mathbf{D}. \quad (155)$$

#### 3.8.1 TRL, LRL, TRM, LRM

The LHKM (TRL, LRL) and LHKM (TRM, LRM) calibrations are described in [17]. These calibration algorithms and the associated uncertainty propagation can lead to over-determined linear and quadratic eigenvalue problems. The over-determined non-linear eigenvalue problem is described in appendix I.



### 3.9 Frequency Conversion Calibrations

#### 3.9.1 Scalar Mixer

The scalar mixer calibration is based on one port calibrations including power for each port, see section 3.1.

The directivity  $E_{xx}$ , the reflection tracking  $E_{x\bar{x}}E_{\bar{x}x}$  and the source match term  $E_{\bar{x}\bar{x}}$  are known for each port  $x$  after the one port calibrations. The magnitudes of the tracking terms  $|E_{x\bar{x}}|$  and  $|E_{\bar{x}x}|$  are known for each port  $x$  after the power calibrations.

The unknown phase of the symmetry term  $\arg(E_{\bar{x}x})$  is assumed as normal distribution  $(0^\circ \pm 180^\circ/\sqrt{2})$  for  $k = 2$ . This uncertainty is uncorrelated over frequency and uncorrelated for each port  $x$ .

The following equation describes the phase of the other tracking term

$$\arg(E_{x\bar{x}}) = \arg(E_{x\bar{x}}E_{\bar{x}x}) - \arg(E_{\bar{x}x}). \quad (156)$$

This yields in an expanded uncertainty for the unknown phase of  $\pm 180^\circ$  for a transmission measurement and  $\pm 0^\circ$  for a reflection measurement. This uncertainty is uncorrelated over frequency.

#### 3.9.2 Unknown Mixer Phase

The unknown mixer phase calibration is based on the scalar mixer calibration except the phase of the symmetry term  $\arg(E_{\bar{x}x})$ , see section 3.9.1.

The absolute phase of the symmetry term is given by

$$\arg(E_{\bar{x}x}) = \arg(E_{\bar{x}x}^O) + \arg(E_{\bar{x}x}^R). \quad (157)$$

where

$\arg(E_{\bar{x}x}^O)$  is the offset phase which is constant for all frequency points and

$\arg(E_{\bar{x}x}^R)$  is the relative phase between different frequency points.

The relative phase of the symmetry term  $\arg(E_{\bar{x}x}^R)$  is computed using the unknown thru calibration, see section 3.4.

The following equation describes the phase of the other tracking term

$$\arg(E_{x\bar{x}}) = \arg(E_{x\bar{x}}E_{\bar{x}x}) - \arg(E_{\bar{x}x}). \quad (158)$$

#### 3.9.3 Unknown Mixer Mag Phase

The unknown mixer mag phase calibration is based on the unknown mixer phase calibration except the magnitude of the symmetry term  $|E_{\bar{x}x}|$ , see section 3.9.2. A power calibration at each port is not needed.

The magnitude of the symmetry term  $|E_{\bar{x}x}|$  and the relative phase of the symmetry term  $\arg(E_{\bar{x}x}^R)$  are computed using the unknown thru calibration, see section 3.4.

The following equation describes the magnitude and phase of the other tracking term

$$E_{x\bar{x}} = \frac{E_{x\bar{x}}E_{\bar{x}x}}{E_{\bar{x}x}}. \quad (159)$$



## 4 VNA Switched Calibration Model

The Switched VNA Model uses  $N$  error terms matrices  $\overset{x}{\mathbf{E}}$  for an  $N$ -port VNA. One for each switch position  $x$  of the source. The switch terms matrix  $\mathbf{W}$  and the associated drift  $\mathbf{V}$  is set to zero. The Switched VNA Model supports the following calibration types: SOLT, Reflection Normalization, Transmission Normalization and One Path Two Ports.

### 4.1 SOLT Calibration

#### 4.1.1 Reflection

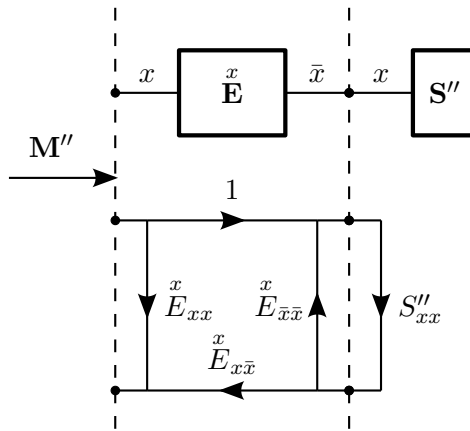


Figure 7: SOLT Reflection Calibration

The following equation describes the cascading of the reflection error terms and the definition of the calibration standard

$$M''_{xx}{}^{(i)} = \overset{x}{E}_{xx} + \frac{\overset{x}{E}_{x\bar{x}} S''_{xx}{}^{(i)}}{1 - \overset{x}{E}_{\bar{x}\bar{x}} S''_{xx}{}^{(i)}} \quad (160)$$

with

$$\bar{x} = N + x. \quad (161)$$

$N$  is the number of ports and  $x$  is the actual port where the reflection calibration is performed. Equation 160 can be rearranged as

$$\underbrace{M''_{xx}{}^{(i)}}_{y_i} = \underbrace{\overset{x}{E}_{xx}}_{p_1} + \underbrace{\overset{x}{E}_{\bar{x}\bar{x}}}_{p_2} M''_{xx}{}^{(i)} S''_{xx}{}^{(i)} + \underbrace{\left( \overset{x}{E}_{x\bar{x}} - \overset{x}{E}_{xx} \overset{x}{E}_{\bar{x}\bar{x}} \right)}_{p_3} S''_{xx}{}^{(i)}. \quad (162)$$

Equation 162 can be written as a system of linear equations

$$\mathbf{p} = \mathbf{A}^{-1} \mathbf{y} \quad (163)$$

with

$$\mathbf{A} = \begin{pmatrix} 1 & M''_{xx}{}^{(1)} S''_{xx}{}^{(1)} & \overset{x}{E}_{\bar{x}\bar{x}} & \overset{x}{E}_{x\bar{x}} - \overset{x}{E}_{xx} \overset{x}{E}_{\bar{x}\bar{x}} \\ 1 & M''_{xx}{}^{(2)} S''_{xx}{}^{(2)} & \overset{x}{E}_{\bar{x}\bar{x}} & \overset{x}{E}_{x\bar{x}} - \overset{x}{E}_{xx} \overset{x}{E}_{\bar{x}\bar{x}} \\ 1 & M''_{xx}{}^{(3)} S''_{xx}{}^{(3)} & \overset{x}{E}_{\bar{x}\bar{x}} & \overset{x}{E}_{x\bar{x}} - \overset{x}{E}_{xx} \overset{x}{E}_{\bar{x}\bar{x}} \end{pmatrix} \quad (164)$$



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and

$$\mathbf{y} = \begin{pmatrix} M_{xx}''(1) \\ M_{xx}''(2) \\ M_{xx}''(3) \end{pmatrix}. \quad (165)$$

For example the first measurement could be a short, the second an open and the third a load. The vector  $\mathbf{p}$  contains the solution for the error terms.

$$\overset{x}{E}_{xx} = p_1 \quad (166)$$

$$\overset{x}{E}_{\bar{x}x} = 1 \quad (167)$$

$$\overset{x}{E}_{x\bar{x}} = p_3 + p_1 p_2 \quad (168)$$

$$\overset{x}{E}_{\bar{x}\bar{x}} = p_2 \quad (169)$$

$\overset{x}{E}_{xx}$  stands for the directivity,  $\overset{x}{E}_{x\bar{x}}\overset{x}{E}_{\bar{x}x}$  denotes the reflection tracking and  $\overset{x}{E}_{\bar{x}\bar{x}}$  designates the source match term.

### 4.1.2 Power

For a power calibration the scaling factor of the tracking terms have to be determined.

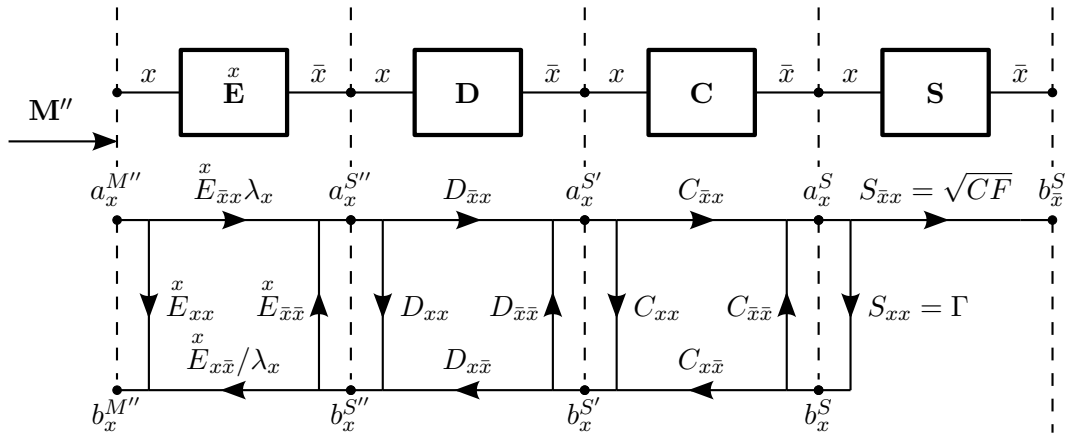


Figure 8: SOLT Power Calibration

In a first step the reflection coefficient  $\Gamma$  of the power sensor is computed using error correction, see section 2.

In a second step the scaling factor  $\lambda_x$  is computed using the receiver values from the VNA, the indicated power of the power sensor, the calibration factor  $CF$  of the power sensor, the error terms and all uncertainty influences.  $\lambda_x$  is computed for each switch position  $x$  of the source. The following set of equations holds

$$b_x^{M''} = a_x^{M''} \overset{x}{E}_{xx} + b_x^{S''} \overset{x}{E}_{x\bar{x}}/\lambda_x \quad (170)$$

$$a_x^{S''} = a_x^{M''} \overset{x}{E}_{\bar{x}x}\lambda_x + b_x^{S''} \overset{x}{E}_{\bar{x}\bar{x}} \quad (171)$$

where



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$a_x^{M''}$  denotes the reference receiver value of port  $x$  including noise and linearity uncertainties,

$b_x^{M''}$  denotes the test receiver value of port  $x$  including noise and linearity uncertainties,

$a_x^{S''}$  denotes the incident wave of the power sensor connected to port  $x$  in  $\sqrt{\text{mW}}$  including drift, cable stability, connector repeatability and standard definition uncertainties,

$b_x^{S''}$  denotes the reflected wave of the power sensor connected to port  $x$  in  $\sqrt{\text{mW}}$  including drift, cable stability, connector repeatability and standard definition uncertainties,

$|b_{\bar{x}}^S|^2$  denotes the indicated power of the power sensor in mW including all absolute and relative uncertainties of the power sensor,

$E_{xx}^x$  denotes the directivity of port  $x$ ,

$E_{x\bar{x}}^x E_{\bar{x}x}^x$  denotes the reflection tracking of port  $x$ ,

$E_{\bar{x}\bar{x}}^x$  denotes the source match term of port  $x$ ,

$\lambda_x$  denotes the scaling factor of the tracking terms for the switch position  $x$  of the source,

$D_{xx}, D_{\bar{x}x}, D_{x\bar{x}}, D_{\bar{x}\bar{x}}$  denote the drift influences of the power sensor measurement at port  $x$ ,

$C_{xx}, C_{\bar{x}x}, C_{x\bar{x}}, C_{\bar{x}\bar{x}}$  denote the cable stability and the connector repeatability influences of the power sensor measurement at port  $x$ ,

$S_{xx}$  denotes the reflection coefficient  $\Gamma$  of the power sensor and

$S_{\bar{x}x}$  denotes the square root of the calibration factor  $\sqrt{CF}$  of the power sensor, see section 6.18.

One can solve equation 170 for  $b_x^{S''}$

$$b_x^{S''} = \frac{b_x^{M''} - a_x^{M''} E_{xx}^x}{E_{x\bar{x}}^x} \lambda_x \quad (172)$$

and plug the result into 171

$$a_x^{S''} = a_x^{M''} E_{\bar{x}x}^x \lambda_x + \frac{b_x^{M''} - a_x^{M''} E_{xx}^x}{E_{x\bar{x}}^x} E_{\bar{x}\bar{x}}^x \lambda_x. \quad (173)$$

One can solve the above equation for the scaling factor  $\lambda_x$  of the tracking terms

$$\lambda_x = \frac{a_x^{S''}}{a_x^{M''} E_{\bar{x}x}^x + \frac{b_x^{M''} - a_x^{M''} E_{xx}^x}{E_{x\bar{x}}^x} E_{\bar{x}\bar{x}}^x}. \quad (174)$$

The phase of  $\lambda_x$  is unknown because the phase of the incident wave of the power sensor  $a_x^{S''}$  is unknown. For further data processing the absolute value of  $\lambda_x$  is used, e.g.: average of multiple power calibrations which finally yields to

$$|\lambda_x| = \sum_{i=1}^n \left| \frac{a_x^{S''(i)}}{a_x^{M''(i)} E_{\bar{x}x}^x + \frac{b_x^{M''(i)} - a_x^{M''(i)} E_{xx}^x}{E_{x\bar{x}}^x} E_{\bar{x}\bar{x}}^x} \right| / n. \quad (175)$$



### 4.1.3 Isolation

The isolation between two ports can be measured directly. Note that it is assumed that there is no drift of the isolation.

$${}^x E_{yx} = M_{yx}^{''(isol)} \tag{176}$$

$${}^y E_{xy} = M_{xy}^{''(isol)} \tag{177}$$

### 4.1.4 Transmission

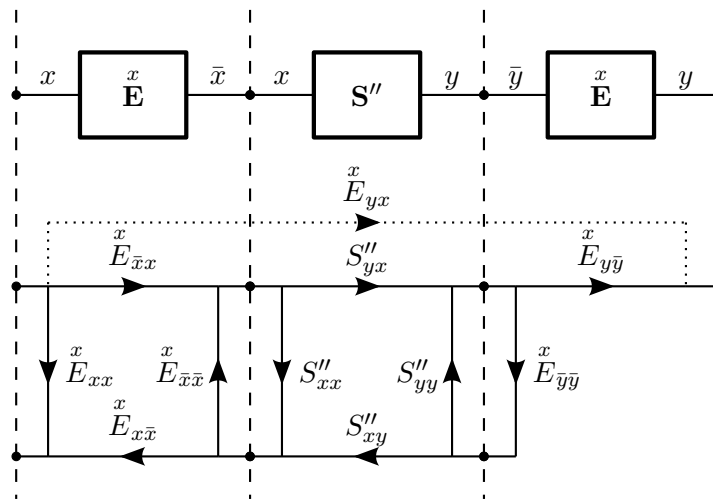


Figure 9: SOLT Forward Transmission Calibration

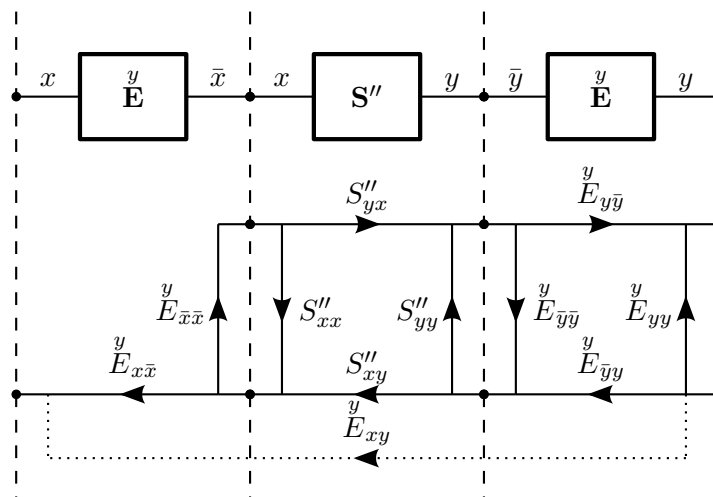


Figure 10: SOLT Reverse Transmission Calibration



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The following S-parameter matrix describes the transmission standard

$$\mathbf{A}^{(i)} = \begin{pmatrix} S_{xx}^{(i)} & S_{xy}^{(i)} \\ S_{yx}^{(i)} & S_{yy}^{(i)} \end{pmatrix}. \quad (178)$$

The error box of port  $x$  is denoted as

$$\mathbf{X} = \begin{pmatrix} E_{xx}^x & E_{x\bar{x}}^x \\ E_{\bar{x}x}^x & E_{\bar{x}\bar{x}}^x \end{pmatrix} \quad (179)$$

with

$$\bar{x} = N + x \quad (180)$$

and the error box of port  $y$  is denoted as

$$\mathbf{Y} = \begin{pmatrix} E_{\bar{y}\bar{y}}^y & E_{\bar{y}y}^y \\ E_{y\bar{y}}^y & E_{yy}^y \end{pmatrix} \quad (181)$$

with

$$\bar{y} = N + y. \quad (182)$$

Cascading the error box of port  $x$  and the thru definition yields a new S-parameter matrix

$$\mathbf{T}^{(i)} = \mathbf{X} \otimes \mathbf{A}^{(i)}. \quad (183)$$

The same can be done for port  $y$

$$\mathbf{T}^{(i)} = \mathbf{A}^{(i)} \otimes \mathbf{Y}. \quad (184)$$

The operator  $\otimes$  denotes the cascading of two 2-ports, as described in appendix A.1.3.

One can introduce four new auxiliary variables, which describe the directivity and isolation corrected measurement of the transmission standard.

$$s_{xx} = M_{xx}^{(thru)} - T_{xx}^{(thru)} \quad (185)$$

$$s_{yx} = M_{yx}^{(thru)} - E_{yx}^x \quad (186)$$

$$s_{xy} = M_{xy}^{(thru)} - E_{xy}^y \quad (187)$$

$$s_{yy} = M_{yy}^{(thru)} - T_{yy}^{(thru)} \quad (188)$$

Next one can come up with equations for the transmission tracking and the load match of port  $x$  and  $y$ .

$$E_{\bar{y}\bar{y}}^x = \frac{s_{xx}}{s_{xx}T_{yy}^{(thru)} + T_{yx}^{(thru)}T_{xy}^{(thru)}} \quad (189)$$

$$E_{y\bar{y}}^x = s_{yx} \frac{1 - E_{\bar{y}\bar{y}}^x T_{yy}^{(thru)}}{T_{yx}^{(thru)}} \quad (190)$$

$$E_{\bar{x}\bar{x}}^y = \frac{s_{yy}}{s_{yy}T_{xx}^{(thru)} + T_{yx}^{(thru)}T_{xy}^{(thru)}} \quad (191)$$

$$E_{x\bar{x}}^y = s_{xy} \frac{1 - E_{\bar{x}\bar{x}}^y T_{yy}^{(thru)}}{T_{xy}^{(thru)}} \quad (192)$$



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$\overline{E}_{yy}^x$  is the load match of port  $y$ ,

$\overline{E}_{yy}^x \overline{E}_{xx}^x$  is the transmission tracking from port  $x$  to port  $y$ ,

$\overline{E}_{xx}^y \overline{E}_{yy}^y$  is the transmission tracking from port  $y$  to port  $x$  and

$\overline{E}_{xx}^y$  is the load match of port  $x$ .



## 4.2 Incomplete Calibrations

### 4.2.1 Reflection Normalization

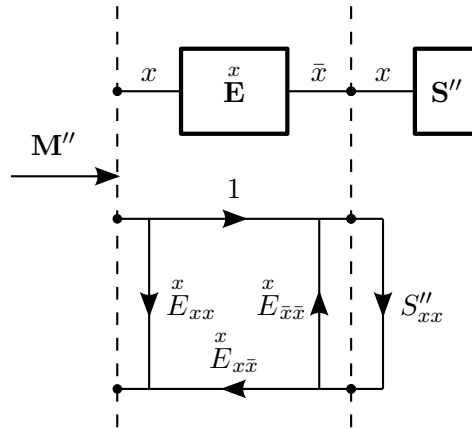


Figure 11: Reflection Normalization

The following equation describes the cascading of the error terms and the definition of the calibration standard

$$M''_{xx} = E_{xx} + \frac{E_{x\bar{x}} S''_{xx}^{(i)}}{1 - E_{\bar{x}\bar{x}} S''_{xx}^{(i)}} \quad (193)$$

with

$$\bar{x} = N + x. \quad (194)$$

$N$  is the number of ports and  $x$  is the actual port where the reflection normalization is performed. Equation 193 can be rearranged as

$$E_{x\bar{x}} = \frac{\left( M''_{xx}^{(i)} - E_{xx} \right) \left( 1 - E_{\bar{x}\bar{x}} S''_{xx}^{(i)} \right)}{S''_{xx}^{(i)}}. \quad (195)$$

Equation 195 is used for reflection normalization where  $E_{x\bar{x}}$  denotes the reflection tracking. The unknown directivity  $E_{xx}$  and the unknown source match  $E_{\bar{x}\bar{x}}$  are assumed as multivariate normal distributions at the origin of the complex plane  $(0 \pm \dots) + (0 \pm \dots)i$ .

For error correction, equation 193 is rearranged as

$$S''_{xx}^{(i)} = \frac{M''_{xx}^{(i)} - E_{xx}}{E_{x\bar{x}} + E_{\bar{x}\bar{x}} \left( M''_{xx}^{(i)} - E_{xx} \right)}. \quad (196)$$



4.2.2 Transmission Normalization

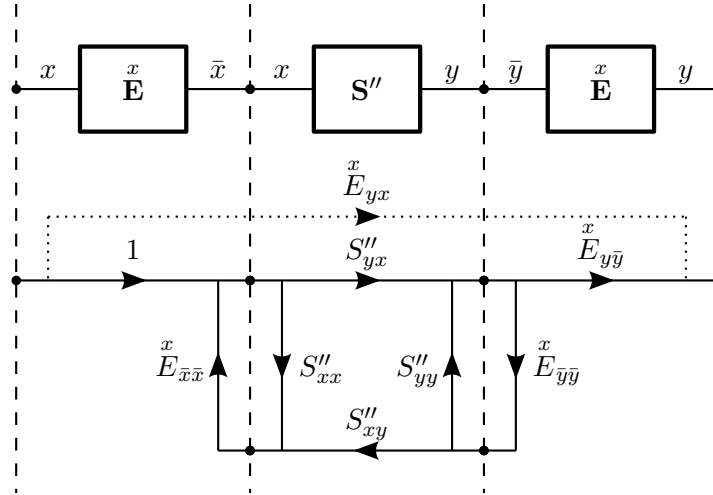


Figure 12: Transmission Normalization

The following equations describe the cascading of the transmission tracking  $E_{y\bar{y}}^x$ , the load match  $E_{\bar{y}\bar{y}}^x$  and the definition of the transmission calibration standard

$$T_{xx}''^{(i)} = S_{xx}''^{(i)} + \frac{S_{xy}''^{(i)} E_{\bar{y}\bar{y}}^x S_{yx}''^{(i)}}{1 - E_{\bar{y}\bar{y}}^x S_{yy}''^{(i)}} \quad (197)$$

and

$$T_{yx}''^{(i)} = \frac{E_{y\bar{y}}^x S_{yx}''^{(i)}}{1 - E_{\bar{y}\bar{y}}^x S_{yy}''^{(i)}} \quad (198)$$

with

$$\bar{x} = N + x \quad (199)$$

$$\bar{y} = N + y. \quad (200)$$

$N$  is the number of ports and  $x$  is the source port and  $y$  is the receiving port.

In a second step, the cascading of the source match  $E_{\bar{x}\bar{x}}^x$ , the crosstalk  $E_{y\bar{y}}^x$  and the results from equations 197 and 198 are used to compute the raw measured transmission

$$M_{yx}''^{(i)} = E_{yx}^x + \frac{T_{yx}''^{(i)}}{1 - E_{\bar{x}\bar{x}}^x T_{xx}''^{(i)}}. \quad (201)$$

Putting equations 197 and 198 into the above equation 201 yields into

$$M_{yx}''^{(i)} = E_{yx}^x + \frac{E_{y\bar{y}}^x S_{yx}''^{(i)}}{\left(1 - E_{\bar{y}\bar{y}}^x S_{yy}''^{(i)}\right) \left(1 - E_{\bar{x}\bar{x}}^x \left(S_{xx}''^{(i)} + \frac{S_{xy}''^{(i)} E_{\bar{y}\bar{y}}^x S_{yx}''^{(i)}}{1 - E_{\bar{y}\bar{y}}^x S_{yy}''^{(i)}}\right)\right)}. \quad (202)$$



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Equation 202 can be rewritten as

$$M_{yx}^{(i)} = E_{yx}^x + \frac{E_{yy}^x S_{yx}^{(i)}}{1 - E_{yy}^x S_{yy}^{(i)} - E_{xx}^x S_{xx}^{(i)} - E_{yy}^x E_{xx}^x S_{yx}^{(i)} S_{xy}^{(i)} + E_{yy}^x E_{xx}^x S_{yy}^{(i)} S_{xx}^{(i)}}. \quad (203)$$

Equation 203 can be rearranged as

$$E_{yy}^x = \frac{M_{yx}^{(i)} - E_{yx}^x}{S_{yx}^{(i)}} \left( 1 - E_{yy}^x S_{yy}^{(i)} - E_{xx}^x S_{xx}^{(i)} - E_{yy}^x E_{xx}^x S_{yx}^{(i)} S_{xy}^{(i)} + E_{yy}^x E_{xx}^x S_{yy}^{(i)} S_{xx}^{(i)} \right). \quad (204)$$

Equation 204 is used for transmission normalization where  $E_{yy}^x$  denotes the transmission tracking. The unknown crosstalk  $E_{yx}^x$ , the unknown source match  $E_{xx}^x$  and the unknown load match  $E_{yy}^x$  are assumed as multivariate normal distributions at the origin of the complex plane  $(0 \pm \dots) + (0 \pm \dots)i$ .

For error correction, equation 203 is rearranged as

$$S_{yx}^{(i)} = \frac{\left( M_{yx}^{(i)} - E_{yx}^x \right) \left( 1 - E_{yy}^x S_{yy}^{(i)} \right) \left( 1 - E_{xx}^x S_{xx}^{(i)} \right)}{E_{yy}^x + \left( M_{yx}^{(i)} - E_{yx}^x \right) E_{yy}^x E_{xx}^x S_{xy}^{(i)}}. \quad (205)$$

The unknown S-parameters of the DUT  $S_{xx}^{(i)}$ ,  $S_{xy}^{(i)}$  and  $S_{yy}^{(i)}$  in equation 205 are assumed as multivariate normal distributions at the origin of the complex plane  $(0 \pm 1) + (0 \pm 1)i$  for  $k = 2$ .

The uncertainties are computed with linear uncertainty propagation. This leads to a well known problem for linear propagation of uncertainties. Multiplying two quantities with value zero and uncertainties larger than zero yields zero with zero uncertainty. Using Monte-Carlo uncertainty propagation yields a value of zero and an uncertainty of the product of the uncertainties of the two input quantities. This is only valid as long the input quantities are not correlated.

To overcome this problem the linear uncertainty propagation has been extended with the following function

$$(0 \pm u(c)) = \text{multiply0by0}((0 \pm u(a)), (0 \pm u(b))) \quad (206)$$

$$= (0 \pm u(a * b)). \quad (207)$$

The above function is used when multiplying  $E_{yy}^x S_{yy}^{(i)}$  and  $E_{xx}^x S_{xx}^{(i)}$  of equation 205. Like that, the error corrected transmission  $S_{yx}^{(i)}$  of the DUT depends as well on the unknown source match, unknown load match and the unknown S-parameters of the DUT.

The second term  $E_{yy}^x E_{xx}^x S_{xy}^{(i)}$  of the denominator of equation 205 can be neglected because it's very small. This finally yields to

$$S_{yx}^{(i)} \approx \frac{M_{yx}^{(i)} - E_{yx}^x}{E_{yy}^x} \left( 1 - \text{multiply0by0} \left( E_{yy}^x, S_{yy}^{(i)} \right) \right) \left( 1 - \text{multiply0by0} \left( E_{xx}^x, S_{xx}^{(i)} \right) \right). \quad (208)$$



4.2.3 One Path Two Ports Calibration

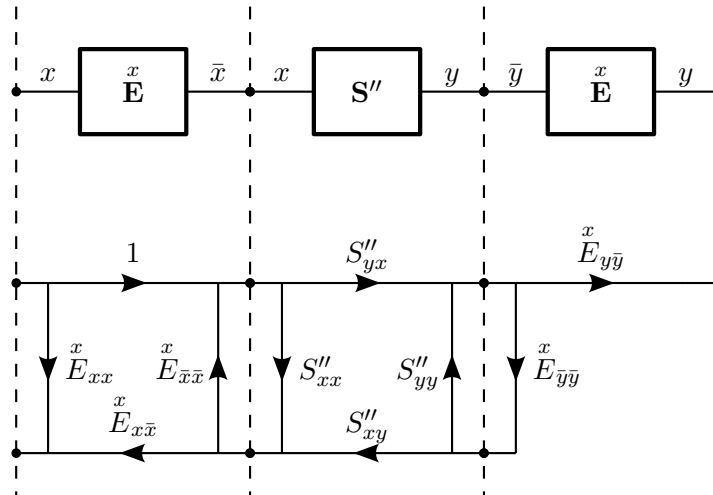


Figure 13: One Path Two Ports Calibration

In a first step, the reflection error terms  $E_{xx}^x$ ,  $E_{x\bar{x}}^x$  and  $E_{\bar{x}\bar{x}}^x$  are computed using a reflection calibration, see section 4.1.1.

In a second step, the the transmission error terms  $E_{yy}^y$  and  $E_{\bar{y}\bar{y}}^x$  are computed using a forward transmission calibration, see section 4.1.4. The S-parameters of the transmission standard definition  $S_{xx}^{(i)}$ ,  $S_{yx}^{(i)}$ ,  $S_{xy}^{(i)}$  and  $S_{yy}^{(i)}$  need to be fully known.

For error correction, see appendix B.3. The unknown reverse S-parameters of the DUT  $S_{xy}^{(i)}$  and  $S_{yy}^{(i)}$  are assumed as multivariate normal distributions at the origin of the complex plane  $(0 \pm 1) + (0 \pm 1)i$  for  $k = 2$ .



## 5 VNA Optimization Calibration

For the optimization calibration [19] the optimizer computes the switch and calibration error terms and the unknown terms of the calibration standard definitions for an over-determined calibration. It uses an optimization algorithm for the VNA measurement model. The following equation describes the in METAS VNA Tools used  $N$ -port VNA measurement model, see section 2.

$$\mathbf{M}^{(i)} = \mathbf{R}^{(i)} + \left[ \left( \mathbf{W} + \mathbf{V}^{(i)} \right) \oplus \left[ \mathbf{E} \oplus \left[ \mathbf{D}^{(i)} \oplus \left[ \mathbf{C}^{(i)} \oplus \mathbf{S}^{(i)} \right] \right] \right] \right] \quad (209)$$

The inverse function of the above equation can be used for error correction.

$$\mathbf{S}^{(i)} = \left[ \left[ \left[ \left( \mathbf{M}^{(i)} - \mathbf{R}^{(i)} \right) \ominus \left( \mathbf{W} + \mathbf{V}^{(i)} \right) \right] \ominus \mathbf{E} \right] \ominus \mathbf{D}^{(i)} \right] \ominus \mathbf{C}^{(i)} \quad (210)$$

The optimizer minimizes the following objective function for all measurements.

$$\left[ \left[ \left[ \left( \mathbf{M}^{(i)} - \mathbf{R}^{(i)} \right) \ominus \left( \mathbf{W} + \mathbf{V}^{(i)} \right) \right] \ominus \mathbf{E} \right] \ominus \mathbf{D}^{(i)} \right] \ominus \mathbf{C}^{(i)} - \mathbf{S}^{(i)} \quad (211)$$

### 5.1 Weighting

The following equation describes the objective function  $f$  where  $\mathbf{X}$  are the variable optimization parameters and  $\mathbf{P}$  are the constant optimization parameters.

$$\mathbf{F} = f(\mathbf{X}, \mathbf{P}) \quad (212)$$

#### 5.1.1 Covariance Weighting

For the weighting of the optimization problem the covariance of the objective function can be used.

$$\mathbf{C}_F = \mathbf{J}_{F,P} \mathbf{C}_P \mathbf{J}'_{F,P} \quad (213)$$

The optimization problem is described with the following expression.

$$\min_{\mathbf{X} \in \mathbb{R}^n} (\mathbf{F} \mathbf{C}_F^{-1} \mathbf{F}') \quad (214)$$

One can introduce  $\mathbf{G} = \mathbf{F} \mathbf{W}_F$ , then the optimization problem becomes

$$\min_{\mathbf{X} \in \mathbb{R}^n} (\mathbf{G} \mathbf{G}') \quad (215)$$

where the weights  $\mathbf{W}_F$  are computed from the covariance of the objective function  $\mathbf{C}_F$  using the Cholesky decomposition and the inverse of a triangular matrix.

#### 5.1.2 User-Defined Weighting

Using user-defined weights  $\mathbf{W}_U$  changes the objective function to

$$\mathbf{G} = \mathbf{F} \mathbf{W}_U. \quad (216)$$

## 5.2 Uncertainty Propagation

The Jacobi matrix  $\mathbf{X}$  to  $\mathbf{P}$  at the point of the solution is described with the following equation.

$$\mathbf{J}_{X,P} = \left( \mathbf{J}'_{G,X} \mathbf{J}_{G,X} \right)^{-1} \mathbf{J}'_{G,X} \mathbf{J}_{G,P} \quad (217)$$

It can be used for the uncertainty propagation.



## 6 VNA Calibration Standard

### 6.1 Agilent Model Standard

The Agilent model standard is described in [20].

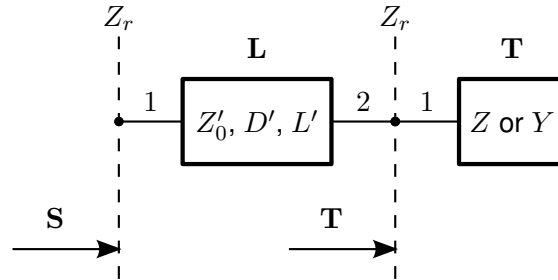


Figure 14: Agilent Model Standard

The S-parameters of an Agilent model reflection standard are defined by the following equation where **L** is the transmission line section and **T** is the reflection part.

$$\mathbf{S} = \mathbf{L} \oplus \mathbf{T} \quad (218)$$

The coaxial transmission line section is computed with the following two equations. Where  $Z'_0$  is the Offset Z0 in Ohm,  $D'$  is the Offset Delay in s and  $L'$  is the Offset Loss in Ohm/s.

$$Z_0 = Z'_0 \left( 1 + (1 - j) \frac{L'}{2\omega Z'_0} \sqrt{\frac{f}{1 \text{ GHz}}} \right) \quad (219)$$

$$\gamma l = j\omega D' \left( 1 + (1 - j) \frac{L'}{2\omega Z'_0} \sqrt{\frac{f}{1 \text{ GHz}}} \right) \quad (220)$$

The waveguide transmission line section is computed with the following equation. Where  $\mu_0$  is the vacuum permeability,  $\epsilon_0$  is the vacuum permittivity,  $f_c$  is the cutoff frequency and  $h/w$  is the height to width ratio.

$$\gamma l = D' \left( L' \sqrt{\frac{\epsilon_0 \epsilon_r}{\mu_0}} \sqrt{\frac{f}{f_c}} \left( \frac{1 + \frac{2h}{w} \left(\frac{f_c}{f}\right)^2}{\sqrt{1 - \left(\frac{f_c}{f}\right)^2}} \right) + j2\pi f \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \right) \quad (221)$$

For computing the S-parameters of a transmission line section see section A.4. Where  $Z_r$  is the reference impedance,  $Z_0$  is the characteristic impedance and  $\gamma l$  is the propagation constant times the length.

#### 6.1.1 Short

The reflection part of a short standard is defined by the following equations.

$$L_{eff} = L_0 + L_1 f + L_2 f^2 + L_3 f^3 \quad (222)$$



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$$Z = j\omega L_{eff} \quad (223)$$

$$T_{11} = \frac{Z - Z_r}{Z + Z_r} \quad (224)$$

For an offset short the transmission line section is cascaded to the reflection part.

### 6.1.2 Open

The reflection part of an open standard is defined by the following equations.

$$C_{eff} = C_0 + C_1 f + C_2 f^2 + C_3 f^3 \quad (225)$$

$$Y = j\omega C_{eff} \quad (226)$$

$$T_{11} = \frac{1 - Z_r Y}{1 + Z_r Y} \quad (227)$$

For an offset open the transmission line section is cascaded to the reflection part.

### 6.1.3 Load

In the Agilent model a load has no reflection.

$$T_{11} = 0 \quad (228)$$

### 6.1.4 Delay / Thru

The S-parameters of a Delay / Thru are equal to the S-parameters of the line section.

$$S = L \quad (229)$$

## 6.2 Anritsu and Rohde Schwarz Model Standard

These model standards are similar to the Agilent model standard except the line section. The Offset Z0  $Z'_0$  is set to the reference impedance.

The Offset Length  $D''$  is defined in m.

$$D'' = \frac{D' c}{\sqrt{\epsilon_r}} \quad (230)$$

Where  $D'$  is the Offset Delay in s,  $c = 299792458$  m/s is the speed of light and  $\epsilon_r = 1$  is the relative permittivity.

The Offset Loss  $L''$  is defined in dB/ $\sqrt{\text{GHz}}$ .

$$L'' = 8.6859 \frac{D' L'}{Z'_0} \quad (231)$$

Where  $Z'_0$  is the Offset Z0 in Ohm,  $D'$  is the Offset Delay in s and  $L'$  is the Offset Loss in Ohm/s.



### 6.3 Databased Standard

The S-parameters of a databased standard are explicitly stated for each data point.

### 6.4 Ideal Standard

Table 2 shows the S-parameters for ideal standards.

Table 2: Ideal Standards

Standard	S-parameter
Ideal Short	$\mathbf{S} = \begin{pmatrix} -1 \end{pmatrix}$
Ideal Open	$\mathbf{S} = \begin{pmatrix} 1 \end{pmatrix}$
Ideal Load	$\mathbf{S} = \begin{pmatrix} 0 \end{pmatrix}$
Ideal Thru	$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
Ideal Isolation	$\mathbf{S} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

### 6.5 Resistance and Series Inductance

The S-parameters of a resistance  $R$  and a series inductance  $L$  are defined by the following equation

$$\mathbf{S} = \begin{pmatrix} \frac{R+j\omega L-Z_r}{R+j\omega L+Z_r} \end{pmatrix}. \quad (232)$$

### 6.6 Resistance and Parallel Capacitance

The S-parameters of a resistance  $R$  and a parallel capacitance  $C$  are defined by the following equation

$$\mathbf{S} = \begin{pmatrix} \frac{1-\left(\frac{1}{R}+j\omega C\right)Z_r}{1+\left(\frac{1}{R}+j\omega C\right)Z_r} \end{pmatrix}. \quad (233)$$



### 6.7 Unknown Standard

Table 3 shows the S-parameters for unknown standards.

Table 3: Unknown Standards

Standard	S-parameter
Unknown Reflection	$\mathbf{S} = ( r )$
Unknown Reflection 2	$\mathbf{S} = ( m e^{-j4\pi f l/c} )$ with $ m  \leq 1$ and $l \geq 0$
Unknown Thru	$\mathbf{S} = \begin{pmatrix} r_1 & t \\ t & r_2 \end{pmatrix}$
Unknown Isolation	$\mathbf{S} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$
Unknown Line	$\mathbf{S} = \begin{pmatrix} 0 & e^{-\gamma l} \\ e^{-\gamma l} & 0 \end{pmatrix}$
Unknown Line 2	$\mathbf{S} = \begin{pmatrix} r_1 & e^{-\gamma l} \\ e^{-\gamma l} & r_2 \end{pmatrix}$ with $\gamma l = g_1 \sqrt{\frac{f}{10^9}} + g_2 \frac{f}{10^9}$
Unknown Series Inductance	$\mathbf{S} = \left( \frac{R+j\omega L-Z_r}{R+j\omega L+Z_r} \right)$ with $L = ?$
Unknown Capacitance	$\mathbf{S} = \left( \frac{1-j\omega C Z_r}{1+j\omega C Z_r} \right)$ with $C = ?$ and $C \geq 0$
Unknown Mixer Mag Phase	$\mathbf{S} = \begin{pmatrix} r_1 & m e^{jp} \\ m e^{jp} & r_2 \end{pmatrix}$
Unknown Mixer Phase	$\mathbf{S} = \begin{pmatrix} r_1 & m_{21} e^{jp} \\ m_{12} e^{jp} & r_2 \end{pmatrix}$



### 6.8 Primary Airline Standard

The S-parameters of a primary airline standard are defined by the following equation

$$\mathbf{S} = \mathbf{O} \oplus \mathbf{K}_1 \oplus \mathbf{C}_1 \oplus \mathbf{L}_1 \oplus \mathbf{P} \oplus \mathbf{L}_2 \oplus \mathbf{C}_2 \oplus \mathbf{K}_2 \oplus \mathbf{O}^{\ominus 1}. \quad (234)$$

$\mathbf{O}$  denotes the line shift, see section 6.8.1.

$\mathbf{K}_1$  denotes the kapton or adapter effect on port 1.

$\mathbf{C}_1$  denotes the half connector of the standard at port 1.

$\mathbf{L}_1$  denotes the half of the line section on port 1 side.

$\mathbf{P}$  denotes a section of ideal line, see section 6.8.2.

$\mathbf{L}_2$  denotes the half of the line section on port 2 side.

$\mathbf{C}_2$  denotes the half connector of the standard at port 2.

$\mathbf{K}_2$  denotes the kapton or adapter effect on port 2.

#### 6.8.1 Line Shift

The S-parameters of the line shift are defined by the following equation

$$\mathbf{O} = \begin{pmatrix} 0 & e^{-j\frac{2\pi f}{c_0}l_{shift}} \\ e^{-j\frac{2\pi f}{c_0}l_{shift}} & 0 \end{pmatrix} \quad (235)$$

where  $f$  is the frequency,  $c_0$  is the speed of light and  $l_{shift}$  is the shift length. The line shift section has to be used when the center conductor of the device of test protrudes into a test port.

If the reference plane at port 1 of the standard definition is behind the calibration reference plane (this means towards the VNA) then the value is negative. For the opposite shift it would be positive and for no shift it is zero.

#### 6.8.2 Propagation Constant

The following equation describes the propagation constant

$$\gamma l = g_1 \sqrt{\frac{f}{1 \text{ GHz}}} + g_2 \frac{f}{1 \text{ GHz}}. \quad (236)$$

$g_1, g_2$  are the unknown parameters and  $f$  is the frequency. The resulting line section is defined by the following equation

$$\mathbf{P} = \begin{pmatrix} 0 & e^{-\gamma l} \\ e^{-\gamma l} & 0 \end{pmatrix}. \quad (237)$$



### 6.9 Primary Offset Short Standard

The S-parameters of a primary offset short standard are defined by the following equation

$$\mathbf{S} = \mathbf{K}_1 \oplus \mathbf{C}_1 \oplus \mathbf{L}_1 \oplus \mathbf{P} \oplus \mathbf{L}_2 \oplus \mathbf{T}. \quad (238)$$

$\mathbf{K}_1$  denotes the kapton or adapter effect on port 1.

$\mathbf{C}_1$  denotes the half connector of the standard.

$\mathbf{L}_1$  denotes the half of the line section on the connector side.

$\mathbf{P}$  denotes a section of ideal line, see section 6.8.2.

$\mathbf{L}_2$  denotes the half of the line section on the side of the short plane.

$\mathbf{T}$  denotes the short plane, see section 6.9.1.

#### 6.9.1 Short Plane

The short plane is defined by the following equations

$$Z = z_1 + z_2 \sqrt{\frac{f}{1 \text{ GHz}}} + z_3 \frac{f}{1 \text{ GHz}} \quad (239)$$

$$T_{11} = \frac{Z - Z_r}{Z + Z_r} \quad (240)$$

where  $z_1, z_2, z_3$  are the unknown parameters,  $f$  is the frequency and  $Z_r$  is the reference impedance.

### 6.10 Primary Flush Short Standard

The S-parameters of a primary flush short standard are defined by the following equation

$$\mathbf{S} = \mathbf{C}_1 \oplus \mathbf{T}. \quad (241)$$

$\mathbf{C}_1$  denotes the half connector of the standard.

$\mathbf{T}$  denotes the short plane, see section 6.9.1.



### 6.11 Waveguide Shim Standard

The S-parameters of a waveguide shim standard are defined by the following equation

$$\mathbf{S} = \mathbf{O}_1 \oplus \mathbf{C}_1 \oplus \mathbf{D} \oplus \mathbf{C}_2 \oplus \mathbf{O}_2. \quad (242)$$

$\mathbf{O}_1$  denotes the effect related to the offset (vertical and horizontal) of the waveguide connector at port 1, see section 6.11.2.

$\mathbf{C}_1$  denotes the half connector of the standard at port 1, see section 6.11.1.

$\mathbf{D}$  denotes the shim, see section 6.11.3.

$\mathbf{C}_2$  denotes the half connector of the standard at port 2, see section 6.11.1.

$\mathbf{O}_2$  denotes the effect related to the offset (vertical and horizontal) of the waveguide connector at port 2, see section 6.11.2.

#### 6.11.1 Waveguide Connector

The S-parameters of the waveguide connector are computed using a transmission line junction, see appendix A.3. The impedances of the test port  $Z_1$  and of the calibration standard  $Z_2$  are defined by the following equations

$$Z_1 = j \frac{2\pi f \mu_1}{\gamma_1} \quad (243)$$

$$Z_2 = j \frac{2\pi f \mu_2}{\gamma_2} \quad (244)$$

where  $f$  is the frequency,  $\mu$  the permeability and  $\gamma$  the propagation constant of the waveguide section, see section 6.11.4.

#### 6.11.2 Waveguide Connector Offset

The S-parameters of the offset (width and height) of the waveguide connector are based on a look up database which has been computed using COMSOL. The following limitations exist:

- The offset in direction of the width has to be between 0 % and 3.1496 % (corresponds to 80  $\mu\text{m}$  in WR10) of the nominal width.
- The offset in direction of the height has to be between 0 % and 6.2992 % (corresponds to 80  $\mu\text{m}$  in WR10) of the nominal height.
- The width height ratio has to be between 2 and 2.5.

#### 6.11.3 Shim

The shim section is defined by the following equation

$$\mathbf{D} = \begin{pmatrix} 0 & e^{-\gamma l} \\ e^{-\gamma l} & 0 \end{pmatrix}. \quad (245)$$

where  $l$  is the length of the shim section and  $\gamma$  is the propagation constant, see section 6.11.4. The unknown parameters in an optimization calibration are the length  $l$  and the conductivity  $\sigma_{DC}$ , which is used to compute the propagation constant  $\gamma$ .



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### 6.11.4 Gamma

The propagation constant of a rectangular waveguide is described in [20]. The following equations are needed:

$$\sigma = \sigma_{DC} - \sigma_{HF} \sqrt{\frac{f}{1 \text{ GHz}}} \quad (246)$$

$$w_e = w - \frac{(4 - \pi) r^2}{h} \quad (247)$$

$$x_0 = \left( \frac{c_0}{2w_e f \sqrt{\epsilon_r}} \right)^2 \quad (248)$$

$$x_1 = \frac{2\pi f \sqrt{\epsilon_r}}{c_0} \quad (249)$$

$$x_2 = \sqrt{1 - x_0} \quad (250)$$

$$x_3 = \frac{\sqrt{\frac{\pi f \mu_0}{\sigma}}}{h} \quad (251)$$

$$x_4 = \sqrt{\frac{\epsilon_0 \epsilon_r}{\mu_0}} \quad (252)$$

$$x_5 = 1 + \frac{2h}{w_e} x_0 \quad (253)$$

$$\gamma = \frac{x_3 x_4 x_5}{x_2} + j x_1 x_2. \quad (254)$$

$\mu$  and  $\epsilon$  are permeability and permittivity. The waveguide section is characterized by his conductivity  $\sigma_{DC}$  and  $\sigma_{HF}$ . The frequency is  $f$ . Width  $w$ , height  $h$  and radius  $r$  describe the geometry of the waveguide section.

### 6.12 Waveguide Offset Short Standard

The S-parameters of a waveguide offset short standard are defined by the following equation

$$\mathbf{S} = \mathbf{O}_1 \oplus \mathbf{C}_1 \oplus \mathbf{D} \oplus \mathbf{T}. \quad (255)$$

$\mathbf{O}_1$  denotes the effect related to the offset (vertical and horizontal) of the waveguide connector at port 1, see section 6.11.2.

$\mathbf{C}_1$  denotes the half connector of the standard, see section 6.11.1.

$\mathbf{D}$  denotes the shim, see section 6.11.3.

$\mathbf{T}$  denotes the short plane, see section 6.12.1.

#### 6.12.1 Short Plane

The short plane is defined by the following equations

$$Z = z_1 + z_2 \sqrt{\frac{f}{1 \text{ GHz}}} + z_3 \frac{f}{1 \text{ GHz}} \quad (256)$$



$$T_{11} = \frac{Z - Z_r}{Z + Z_r} \quad (257)$$

where  $z_1, z_2, z_3$  are the unknown parameters,  $f$  is the frequency and  $Z_r$  is the reference impedance.

### 6.13 Waveguide Flush Short Standard

The S-parameters of a waveguide flush short standard are defined by the following equation

$$\mathbf{S} = \mathbf{T}. \quad (258)$$

$\mathbf{T}$  denotes the short plane, see section 6.12.1.

### 6.14 Simple Line Standard

The distributed admittance is computed with the following equation

$$Y' = G' + j\omega C' \quad (259)$$

where  $G'$  and  $C'$  are specified by the user per frequency point. The transmission line section is computed with the following two equations

$$\gamma l = \left( x_1 \sqrt{\epsilon_r} + jx_2 \frac{\omega}{c_0} \sqrt{\epsilon_r} \right) l \quad (260)$$

$$Z_0 = \frac{\gamma}{Y'} \quad (261)$$

where  $l$  is the length of the line,  $c_0$  is the speed of light,  $\epsilon_r$  is the relative permittivity and  $x_1, x_2$  are the unknown parameters for each frequency. For computing the S-parameters of a transmission line section see section A.4.

### 6.15 On-Wafer Line Standard

The start values for the characteristic impedance  $Z'_0$  and the propagation constant  $\gamma'$  are computed using the on-wafer line model which is described in [21], [22], [23], [24], [25], [26] and [27].

After computing the start values for the characteristic impedance  $Z'_0$  and the propagation constant  $\gamma'$  the distributed admittance is computed with the following equation.

$$Y' = \frac{\gamma'}{Z'_0} \quad (262)$$

The transmission line section is computed with the following two equations

$$\gamma l = (x_1 + \text{Re}(\gamma') + jx_2 \text{Im}(\gamma')) l \quad (263)$$

$$Z_0 = \frac{\gamma}{Y'} \quad (264)$$

where  $l$  is the length of the line and  $x_1, x_2$  are the unknown parameters for each frequency. For computing the S-parameters of a transmission line section see section A.4.

There are different versions of the on-wafer line model:



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**Heinrich Paper 1993** is the implementation of the Heinrich model described in [21].

**FORTTRAN Code 1993** is the implementation of the modifications described in [22] which is based on the Heinrich model [21]. The equations 8, 17 and 18 from the Heinrich model [21] are modified according to the 1993 version of his FORTRAN code [22], which take into account results of his work obtained after the paper [21] went to print.

$$\omega_{c1} := \omega_{c1} \sqrt{2} \quad (265)$$

$$F_L^{(c)} := F_L^{(c)} \left( 1 - 0.1 \left( \frac{a}{b} \right)^2 \right) \quad (266)$$

$$F_L^{(g)} := F_L^{(g)} \left( 1 - 0.1 \left( \frac{a}{b} \right)^2 \right) \quad (267)$$

**Schnieder Paper 2003** is the implementation of the modifications described in [23] which is based on the modified Heinrich model [22]. The following equations are based on the equations 1, 2 and 3 from the modifications described in [23].

$$w_{tot} = w + 2s + 2w_g \quad (268)$$

$$f_g = \frac{2c_0}{w_{tot} \sqrt{2(\epsilon_r - 1)}} \quad (269)$$

$$d = w + 2s \quad (270)$$

$$p = \frac{2.86465 \left( \frac{d}{w_{tot}} \right)^2}{0.15075 + \frac{d}{w_{tot}}} \quad (271)$$

The correction  $C_{corr}$  for the capacitance  $C$  is computed based on the FORTRAN code from Schnieder and not using equation 4 from the paper [23].

$$C_{er} = 2\epsilon_0\epsilon_r(F_{up} + F_{low}) \quad (272)$$

$$C_q = 2\epsilon_0(1 + \epsilon_r)F_{low} \quad (273)$$

$$C_{factor} = 1 + \left( \sqrt{\frac{C_{er}}{C_q}} - 1 \right) p \left( \frac{f}{f_g} \right)^2 \quad (274)$$

$$C_{corr} = C(C_{factor} - 1) \quad (275)$$

$$C := C + C_{corr} \quad (276)$$



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The correction  $R_{corr}$  for the resistance  $R$  is computed based on the equation 16 from the Schnieder paper [23].

$$R_{corr} = R_{rad}(w, s, w_g, t, f, \epsilon_r) \quad (277)$$

$$R := R + R_{corr} \quad (278)$$

**Gold Paper 2019** is the implementation of the modifications described in [24], [25] and [26] which is based on the modified Schnieder model [23] and adds surface roughness to the model. The following ordinary differential equations are used

$$\frac{\partial y_1}{\partial x} = y_2 \quad (279)$$

$$\frac{\partial y_2}{\partial x} = \frac{\partial \ln \sigma(x)}{\partial x} y_2 - \omega \mu_0 \sigma(x) y_3 \quad (280)$$

$$\frac{\partial y_3}{\partial x} = y_4 \quad (281)$$

$$\frac{\partial y_4}{\partial x} = \frac{\partial \ln \sigma(x)}{\partial x} y_4 + \omega \mu_0 \sigma(x) y_1 \quad (282)$$

with

$$\sigma(x) = \sigma_{bulk} \text{NormCDF}(x, 0, R_q) \quad (283)$$

and

$$\frac{\partial \ln \sigma(x)}{\partial x} = \frac{\text{NormPDF}(x, 0, R_q)}{\text{NormCDF}(x, 0, R_q)} \quad (284)$$

where  $R_q$  is the surface roughness in m. The above ODE system is derived from equation 25 in [25].

The ODE system is solved backwards with  $n = 2001$  points and a relative epsilon of  $10^{-9}$  over the following interval

$$x_{start} = \text{NormInvCDF}\left(\frac{\omega \epsilon_r}{\mu_0 c_0^2} \sigma_{bulk}, 0, 1\right) R_q \quad (285)$$

$$x_{stop} = 15 \frac{2}{\sqrt{\omega \mu_0 \sigma_{bulk}}} \quad (286)$$

with the initial conditions  $y_{n,1} = 10^{-6}$ ,  $y_{n,2} = 0$ ,  $y_{n,3} = 0$  and  $y_{n,4} = 0$ .

The normalized magnetic field  $B_y$  and the derivative  $\frac{\partial B_y}{\partial x}$  are computed with the following equations.

$$B_{y,i} = \frac{y_{i,1} + j y_{i,3}}{y_{1,1} + j y_{1,3}} \quad (287)$$

$$\frac{\partial B_{y,i}}{\partial x} = \frac{y_{i,2} + j y_{i,4}}{y_{1,1} + j y_{1,3}} \quad (288)$$

The current density  $J_z$  and the loss power density  $p_V$  is computed based on equations 27 and 28 in [25].

$$J_z = \frac{1}{\mu_0} \frac{\partial B_y}{\partial x} \quad (289)$$



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$$p_V = \frac{|J_z|^2}{2\sigma(x)} \quad (290)$$

The effective conductivity is computed with the following integral based on equation 31 in [25].

$$\sigma_{\text{eff}} = \frac{\omega}{8\mu_0^3} |B_{y,1}|^4 \left[ \int_{x_{\text{start}}}^{x_{\text{stop}}} p_V dx \right]^{-2} \quad (291)$$

The effective permeability is computed with the following integral based on equation 33 in [25].

$$\mu_{r,\text{eff}} = 2\mu_0\sigma_{\text{bulk}}\omega |B_{y,1}|^{-4} \left[ \int_{x_{\text{start}}}^{x_{\text{stop}}} |B_y|^2 dx \right]^2 \quad (292)$$

Finally the equations 6, 7, 9 and the expressions  $R'_{c1}$ ,  $R'_{g1}$ ,  $L'_{z2}$  of the Heinrich model [21] are modified using equations 3, 4, 5, 6, 7 and 8 in [26].

**Phung Paper 2021** is the implementation of the modifications described in [27] which is based on the modified Schnieder model [23]. The following equations are based on the equations 3, 4, 5 and 6 from the modifications described in [27].

$$f_1 = 1 + 0.45 \left( \frac{w}{d} \right)^4 \quad (293)$$

$$f_2 = 1.87 + \frac{273.18}{47.6 + 1.29(\epsilon_r - 9)^2} \quad (294)$$

$$f_3 = \frac{w_g}{w_{\text{tot}}} \quad (295)$$

$$f_{g1} = \frac{2c_0}{1.1w_{\text{tot}}\sqrt{2(\epsilon_r - 1)}} \quad (296)$$

The correction  $R_{\text{corr}}$  for the resistance  $R$  is computed based on the FORTRAN code from Phung and not using equation 2 from the paper [27].

$$R_{\text{corr}} := R_{\text{corr}} \left( 1 + \frac{f_1 f_2 f_3}{1 + 19.83 \left( \frac{f}{f_{g1}} - 1 \right)^2} \right) \quad (297)$$

$$R := R + R_{\text{corr}} \quad (298)$$

### 6.16 On-Wafer Offset Short Standard

The S-parameters of an on-wafer offset short standard are defined by the following equation

$$\mathbf{S} = \mathbf{L} \oplus \mathbf{T}. \quad (299)$$

$\mathbf{L}$  denotes the offset line section, see section 6.16.1.

$\mathbf{T}$  denotes the short plane, see section 6.16.2.



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### 6.16.1 Offset Line Section

The characteristic impedance  $Z_0$  and the propagation constant  $\gamma$  of the offset line section are computed using the Heinrich model which is described in [21]. The equations 8, 17 and 18 from the Heinrich model [21] are modified, see section 6.15. The length  $l$  is the unknown parameter. For computing the S-parameters of a transmission line section see section A.4.

### 6.16.2 Short Plane

The short plane is defined by the following equations

$$Z = z_1 + z_2 \sqrt{\frac{f}{1 \text{ GHz}}} + z_3 \frac{f}{1 \text{ GHz}} \quad (300)$$

$$T_{11} = \frac{Z - Z_r}{Z + Z_r} \quad (301)$$

where  $z_1$ ,  $z_2$ ,  $z_3$  are the unknown parameters,  $f$  is the frequency and  $Z_r$  is the reference impedance.

### 6.17 On-Wafer Flush Short Standard

The S-parameters of an on-wafer flush short standard are defined by the following equation

$$\mathbf{S} = \mathbf{T}. \quad (302)$$

$\mathbf{T}$  denotes the short plane, see section 6.16.2.



### 6.18 Power Sensor Standard

The power sensor standard is described as a two-port using S-parameters, see figure 15

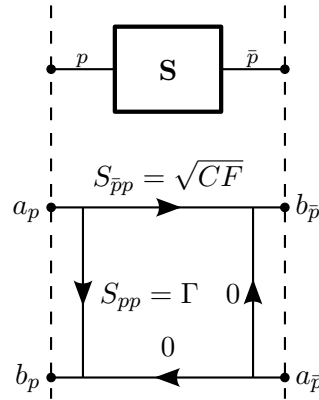


Figure 15: Power sensor standard

where

$S_{pp}$  denotes the reflection coefficient  $\Gamma$  of the power sensor,

$S_{\bar{p}p}$  denotes the square root of the calibration factor  $\sqrt{CF}$  of the power sensor,

$a_p$  denotes the incident wave in  $\sqrt{\text{mW}}$ ,

$b_p$  denotes the reflected wave in  $\sqrt{\text{mW}}$ ,

$|a_p|^2$  denotes the incident power in mW and

$|b_{\bar{p}}|^2$  denotes the indicated power in mW.

The following concept is proposed to map power sensor measurements with the VNA: An  $N$ -port VNA is extended by further  $N$  logical ports. As a difference to a physical port  $p$  the logical port  $\bar{p}$  has no error network associated with it. For measurements, a power sensor, described as a two-port device, is inserted between a physical port and a logical port. Each physical port  $p$  has a corresponding logical port  $N + p$ .



## 7 VNA Uncertainty Contributions

Tables 4, 5, 6, 7 and 8 show the uncertainty input ids.

### 7.1 Noise and linearity

The noise influence is uncorrelated for each measurement because it's a random effect. The linearity influence is correlated for each measurement because it's a systematic effect.

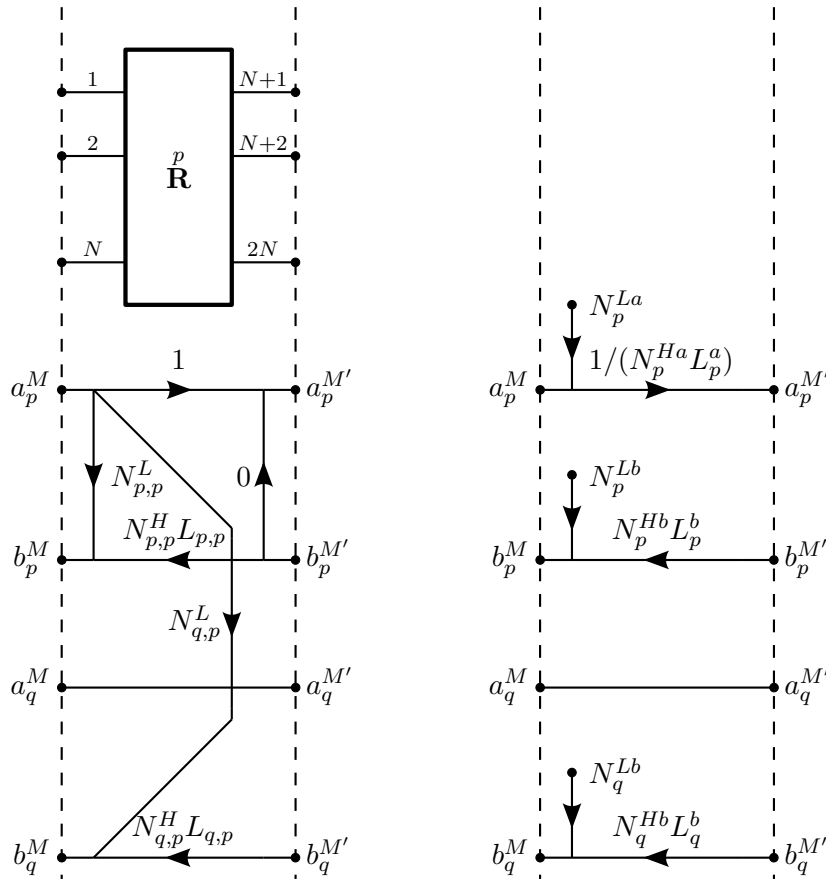


Figure 16: Noise and linearity influences (left: S-parameters, right: receivers)

Uncertainty definition:

$N_{p,p}^L = (b_p^M + N_p^{Lb}) / (a_p^M + N_p^{La}) - b_p^M / a_p^M$  is the noise floor in dB (add.) of source port  $p$ .

$N_{q,p}^L = (b_q^M + N_q^{Lb}) / (a_p^M + N_p^{La}) - b_q^M / a_p^M$  is the noise floor in dB (add.) from port  $p$  to  $q$ .

$N_{p,p}^H = N_p^{Hb} / N_p^{Ha}$  is the trace noise in dB and deg (multiplicative) of source port  $p$ .

$N_{q,p}^H = N_q^{Hb} / N_p^{Ha}$  is the trace noise in dB and deg (multiplicative) from port  $p$  to  $q$ .

$L_{p,p} = L_p^b / L_p^a$  is the linearity in dB and deg (multiplicative) of source port  $p$ .

$L_{q,p} = L_q^b / L_p^a$  is the linearity in dB and deg (multiplicative) from port  $p$  to  $q$ .



### 7.2 Drift of switch and error terms

The structure of  $\mathbf{D}$  can but must not be a copy of  $\mathbf{E}$ . The drift influence  $\mathbf{D}$  is acting on corrected S-parameters. Those the specification of  $\mathbf{D}$  should be for the drift of the corrected S-parameters.

The individual drift contributions for the switch and error terms are uncorrelated for each measurement. The single drift terms are partly correlated over time. E.g.: the drift in directivity of a measurement  $i$  and  $i + 1$  are partly correlated whereas there is no correlation between directivity and tracking drift.

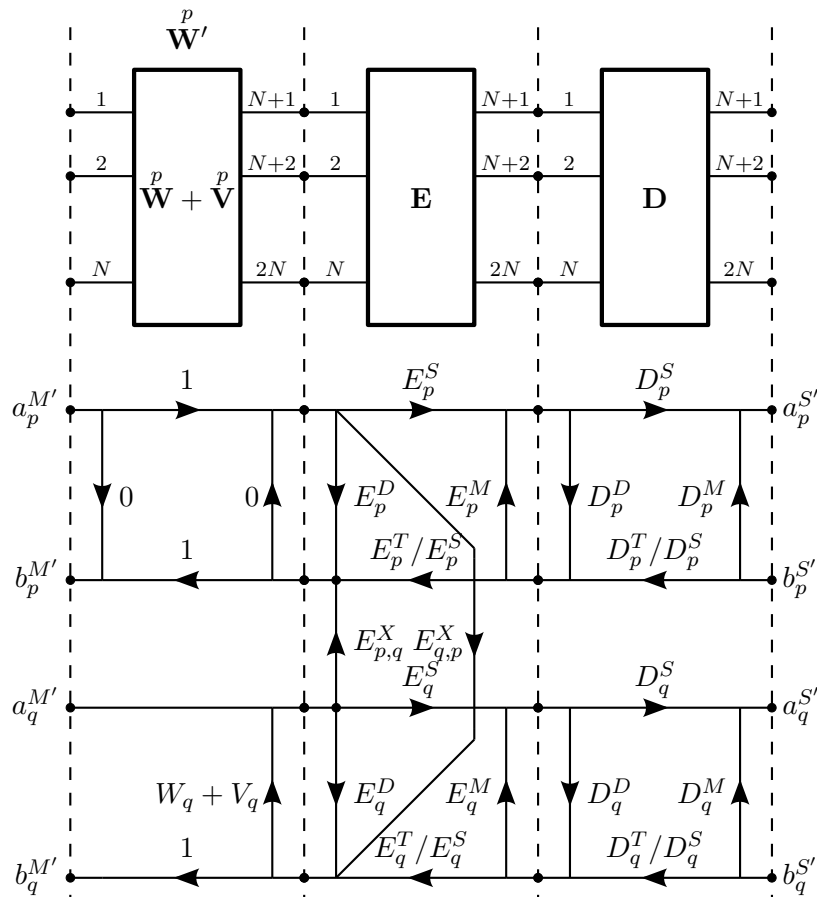


Figure 17: Drift of switch and error terms

Uncertainty definition:

$V_q$  is the switch term drift in dB (additive) of port  $q$  when  $p$  is the source port.

$D_p^D$  is the directivity drift in dB (additive) of port  $p$ .

$D_p^T$  is the tracking drift in dB and deg (multiplicative) of port  $p$ .

$D_p^S$  is the symmetry drift in dB and deg (multiplicative) of port  $p$ .

$D_p^M$  is the match drift in dB (additive) of port  $p$ .



### 7.3 Cable stability, connector repeatability, switchable unit repeatability and DUT uncertainty

The cable influences are uncorrelated for each new cable position. The connector influences are uncorrelated for each new connection. The switchable unit influences are uncorrelated for each new switch state. The DUT influences are uncorrelated for each new DUT index.

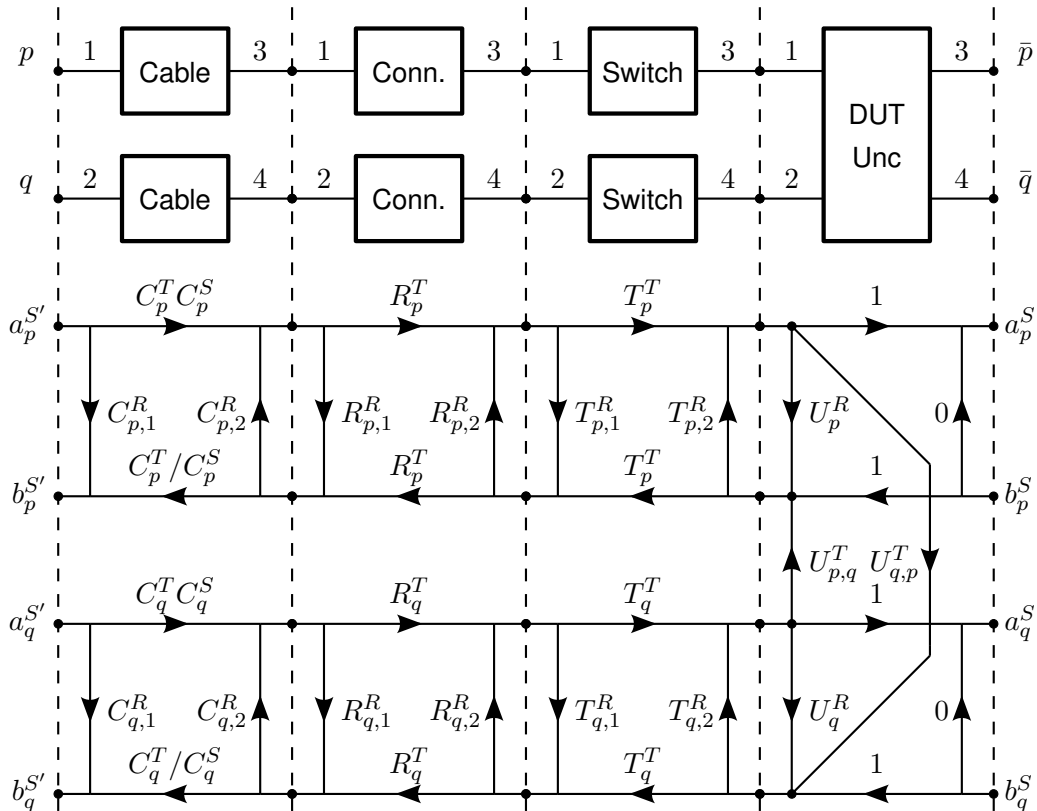


Figure 18: Cable stability, connector repeatability, switchable unit repeatability and DUT uncertainty

Uncertainty definition:

$C_{p,1}^R, C_{p,2}^R$  is the cable reflection stability in dB (additive) of port  $p$ .

$C_p^T$  is the cable transmission stability in dB and deg (multiplicative) of port  $p$ .

$C_p^S$  is the cable symmetry stability in dB and deg (multiplicative) of port  $p$ .

$R_{p,1}^R, R_{p,2}^R$  is the connector reflection repeatability in dB (additive) of port  $p$ .

$R_p^T$  is the connector transmission repeatability in dB and deg (multiplicative) of port  $p$ .

$T_{p,1}^R, T_{p,2}^R$  is the reflection repeatability of a switchable unit in dB (additive) of port  $p$ .

$T_p^T$  is the transmission repeatability of a switchable unit in dB and deg (multiplicative) of port  $p$ .



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$U_p^R$  is the DUT reflection uncertainty (additive) of port  $p$ .

$U_{p,q}^T$  is the DUT transmission uncertainty (additive) from port  $q$  to  $p$ .

A switchable unit can be an electronic calibration unit (ECU), a step attenuator or any other switchable device under test (DUT).

The DUT uncertainty can be used to represent the crosstalk in on-wafer measurements. In this case the DUT uncertainty is uncorrelated for every standard on the wafer and it's correlated for multiple measurements of the same standard.



Table 4: Uncertainty Input IDs

Unc Contribution	Global ID 128 bit	Influence 16 bit	Reserve 8 bit	Version 8 bit	Counter 63–48 bit	47–40 bit	39–32 bit	31–1 bit	0 bit
Unknown	Random ID	0x0000	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
CMC Entry	CMC ID	0x0001	0x00	0x00	Random ID	Rcv Port	Src Port	Freq	RI
CMC Entry	CMC ID	0x0002	0x00	0x00	Random ID	Rcv Port	Src Port	Freq	MP
VNA Exp Statistical	Journal ID	0x0008	0x00	0x00	Meas Count	Rcv Port	Src Port	Freq	RI
	Journal ID	0x0008	0x00	0x01, 0x02	Meas Count	Contribution		Freq	0
	Journal ID	0x0008	0x00	0x03	Meas Count	0x00	Ref Rcv	Freq	0
VNA Exp Systematic	Journal ID	0x0009	0x00	0x00	Exp Count	Rcv Port	Src Port	Freq	RI
	Journal ID	0x0009	0x00	0x01, 0x02	Exp Count	Contribution		Freq	0
	Journal ID	0x0009	0x00	0x03	Exp Count	0x00	Ref Rcv	Freq	0
VNA Type A	Random ID	0x000A	0x00	0x00	0x0000	Contribution		Freq	0
	Random ID	0x000A	0x00	0x01	0x0000	0x00	Ref Rcv	Freq	0
	Journal ID	0x000A	0x00	0x02	Meas Count	Contribution		Freq	0
	Journal ID	0x000A	0x00	0x03	Meas Count	0x00	Ref Rcv	Freq	0
VNA Noise Floor	Journal ID	0x0011	0x00	0x00	Meas Count	Rcv Port	Src Port	Freq	RI
VNA Noise Floor	Journal ID	0x0011	0x00	0x01	Meas Count	Swt Port	0	Freq	RI
VNA Noise Floor	Journal ID	0x0011	0x00	0x02	Meas Count	0	Ref Rcv	Freq	RI
VNA Noise Trace	Journal ID	0x0012	0x00	0x00	Meas Count	Rcv Port	Src Port	Freq	MP
VNA Noise Trace	Journal ID	0x0012	0x00	0x01	Meas Count	Swt Port	0	Freq	MP
VNA Noise Trace	Journal ID	0x0012	0x00	0x02	Meas Count	0	Ref Rcv	Freq	MP
VNA Linearity	VNA ID	0x0020	0x00	0x00	$100(p + 320)$	Rcv Port	0x00	0x00000000	MP
	VNA ID	0x0020	0x00	0x01, 0x02	0x0000	Rcv Port	0x00	$10^6(p + 1000)$	MP
	VNA ID	0x0020	0x00	0x03	0x0000	Swt Port	0x00	$10^6(p + 1000)$	MP
	VNA ID	0x0020	0x00	0x04	0x0000	Ref Rcv	0x00	$10^6(p + 1000)$	MP



Table 5: Uncertainty Input IDs cont.

Unc Contribution	Global ID 128 bit	Influence 16 bit	Reserve 8 bit	Version 8 bit	Counter 63–48 bit	47–40 bit	39–32 bit	31–1 bit	0 bit
VNA Drift Switch Term	Journal ID	0x0031	0x00	0x00, 0x01, 0x02, 0x03	Meas Count	Port	0x00	Freq	RI
VNA Drift Directivity	Journal ID	0x0032	0x00	0x00, 0x01, 0x02, 0x03	Meas Count	Port	0x00	Freq	RI
VNA Drift Tracking	Journal ID	0x0033	0x00	0x00, 0x01, 0x02, 0x03	Meas Count	Port	0x00	Freq	MP
VNA Drift Match	Journal ID	0x0034	0x00	0x00, 0x01, 0x02, 0x03	Meas Count	Port	0x00	Freq	RI
VNA Drift Isolation	Journal ID	0x0035	0x00	0x00, 0x01, 0x02, 0x03	Meas Count	Rcv Port	Src Port	Freq	RI
VNA Drift Symmetry	Journal ID	0x0036	0x00	0x02, 0x03	Meas Count	Port	0x00	Freq	MP
Cable Transmission	Journal ID	0x0040	0x00	0x00	Cable Pos	Port	0x00	Freq	MP
Cable Reflection	Journal ID	0x0041	0x00	0x00	Cable Pos	Port	$C_1^R, C_2^R$	Freq	RI
Cable Symmetry	Journal ID	0x0042	0x00	0x00	Cable Pos	Port	0x00	Freq	MP
Connector Reflection	Journal ID	0x0050	0x00	0x00	Conn Count	Port	$R_1, R_2$	Freq	RI
Connector Transmission	Journal ID	0x0051	0x00	0x00	Conn Count	Port	0x00	Freq	MP
Switchable Unit Refl.	Journal ID	0x0054	0x00	0x00	State Count	Port	$R_1, R_2$	Freq	RI
Switchable Unit Trans.	Journal ID	0x0055	0x00	0x00	State Count	Port	0x00	Freq	MP
DUT Uncertainty	Journal ID	0x0058	0x00	0x00	DUT Index	Rcv Port	Src Port	Freq	RI
Cal Std Short	Cal Std ID	0x0061	0x00	0x00	0x0000	0x00	0x00	0x00000000	MP
Cal Std Open	Cal Std ID	0x0062	0x00	0x00	0x0000	0x00	0x00	0x00000000	MP
Cal Std Load	Cal Std ID	0x0063	0x00	0x00	0x0000	0x00	0x00	0x00000000	RI
Cal Std Thru / Delay Refl.	Cal Std ID	0x0064	0x00	0x00	0x0000	0x00	0x00	0x00000000	RI
Cal Std Short	Cal Std ID	0x0061	0x00	0x01	Frequency (mHz)				MP
Cal Std Open	Cal Std ID	0x0062	0x00	0x01	Frequency (mHz)				MP
Cal Std Load	Cal Std ID	0x0063	0x00	0x01	Frequency (mHz)				RI
Cal Std Thru / Delay Refl.	Cal Std ID	0x0064	0x00	0x01	Frequency (mHz)				RI
Cal Std Thru / Delay Trans.	Cal Std ID	0x0065	0x00	0x01	Frequency (mHz)				MP



Table 6: Uncertainty Input IDs cont.

Unc Contribution	Global ID 128 bit	Influence 16 bit	Reserve 8 bit	Version 8 bit	Counter 63–48 bit	47–40 bit	39–32 bit	31–1 bit	0 bit
Agilent Unc Calculator	Random ID	0x0071	0x00	0x00	0x0000	Rcv Port	Src Port	Freq	RI
Agilent Unc CITI File RI	Data ID	0x0072	0x00	0x00	0x0000	Rcv Port	Src Port	Freq	RI
Agilent Unc CITI File MP	Data ID	0x0073	0x00	0x00	0x0000	Rcv Port	Src Port	Freq	MP
Unknown VNA Directivity	Journal ID	0x0078	0x00	0x00	0x0000	Port	0x00	Freq	RI
Unknown VNA Refl. Tracking	Journal ID	0x0079	0x00	0x00	0x0000	Port	0x00	Freq	MP
Unknown VNA Source Match	Journal ID	0x007A	0x00	0x00	0x0000	Port	0x00	Freq	RI
Unknown VNA Isolation	Journal ID	0x007B	0x00	0x00	0x0000	Rcv Port	Src Port	Freq	RI
Unknown VNA Trans. Tracking	Journal ID	0x007C	0x00	0x00	0x0000	Rcv Port	Src Port	Freq	MP
Unknown VNA Load Match	Journal ID	0x007D	0x00	0x00	0x0000	Rcv Port	Src Port	Freq	RI
Unknown VNA Symmetry	Journal ID	0x007E	0x00	0x00	0x0000	Port	0x00	Freq	MP
	Journal ID	0x007E	0x00	0x01	0x0000	Port	0x00	0x00000000	MP
Unknown S-Parameters	Journal ID	0x0080	0x00	0x00	Meas Count	Rcv Port	Src Port	Freq	RI
Electrical Resistance	Random ID	0x0093	0x00	0x00	0x0000	0x00	0x00	Contribution	
Load DC Resistance	Cal Std ID	0x0094	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Load Fit Error	Cal Std ID	0x0095	0x00	0x00	0x0000	0x00	0x00	0x00000000	RI
Short DC Resistance	Cal Std ID	0x0096	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Series Inductance	Cal Std ID	0x0098	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Parallel Capacitance	Cal Std ID	0x0099	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
TD Unknown DC Point	Random ID	0x0100	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
TD Unknown Frequency Point	Data ID	0x0101	0x00	0x00	Frequency (mHz)				RI
Data Set	Data ID	0x0200	0x00	0x00	Freq			Contribution	



Table 7: Uncertainty Input IDs cont.

Unc Contribution	Global ID 128 bit	Influence 16 bit	Reserve 8 bit	Version 8 bit	Counter 63–48 bit	47–40 bit	39–32 bit	31–1 bit	0 bit
Noise Parameters Physical Temperature	Data ID	0x0280	0x00	0x00	Physical Temperature (mK)			Uncertainty (mK)	
Material Parameter Length	Data ID	0x0300	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Material Relative Permittivity	Cal Std ID	0x0400	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Material Relative Permeability	Cal Std ID	0x0401	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Material Conductivity	Cal Std ID	0x0402	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Material DC Conductivity	Cal Std ID	0x0403	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Material HF Conductivity	Cal Std ID	0x0404	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Material Tan Delta	Cal Std ID	0x0405	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Conn Pin Depth	Cal Std ID	0x0410	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Pin Gap	Cal Std ID	0x0411	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Outer Chamfer	Cal Std ID	0x0412	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Inner Chamfer	Cal Std ID	0x0413	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Male Outer Chamfer	Cal Std ID	0x0414	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Male Inner Chamfer	Cal Std ID	0x0415	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Male Pin Diameter	Cal Std ID	0x0416	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Hole Diameter	Cal Std ID	0x0417	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Outer Conductor Diameter	Cal Std ID	0x0418	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Male Center Conductor Diameter	Cal Std ID	0x0419	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Hole Length	Cal Std ID	0x0420	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Center Conductor Diameter	Cal Std ID	0x0421	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Slot Length	Cal Std ID	0x0430	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Slot Width	Cal Std ID	0x0431	0x00	0x00	Port	0x00	0x00	0x00000000	0
Conn Female Slot Chamfer	Cal Std ID	0x0432	0x00	0x00	Port	0x00	0x00	0x00000000	0



Table 8: Uncertainty Input IDs cont.

Unc Contribution	Global ID 128 bit	Influence 16 bit	Reserve 8 bit	Version 8 bit	Counter 63–48 bit	47–40 bit	39–32 bit	31–1 bit	0 bit
Conn Uncompressed Mid Finger Diameter	Cal Std ID	0x0441	0x00	0x00	Port	0x00	0x00	Pos Index	0
Conn Compressed Mid Finger Diameter	Cal Std ID	0x0442	0x00	0x00	Port	0x00	0x00	Pos Index	0
Conn Outer Diameter in Finger Sections	Cal Std ID	0x0443	0x00	0x00	Port	0x00	0x00	Pos Index	0
Connector FDTD Real	Cal Std ID	0x0450	0x00	0x00	Port	0x00	0x00	0x00000000	0
Connector FDTD Imag	Cal Std ID	0x0451	0x00	0x00	Port	0x00	0x00	0x00000000	0
Pin Gap FDTD Real	Cal Std ID	0x0452	0x00	0x00	Port	0x00	0x00	0x00000000	0
Pin Gap FDTD Imag	Cal Std ID	0x0453	0x00	0x00	Port	0x00	0x00	0x00000000	0
Standard Length	Cal Std ID	0x0460	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Line Length	Cal Std ID	0x0461	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Line z Position	Cal Std ID	0x0462	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Line ICOD	Cal Std ID	0x0463	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Line OCID	Cal Std ID	0x0464	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Line Attenuation Constant	Cal Std ID	0x0470	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Line Phase Constant	Cal Std ID	0x0471	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Short Plane ICOD	Cal Std ID	0x0480	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Short Plane OCID	Cal Std ID	0x0481	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Misc Line Shift	Cal Std ID	0x0490	0x00	0x00	0x0000	0x00	0x00	0x00000000	0



Table 9: Uncertainty Input IDs cont.

Unc Contribution	Global ID 128 bit	Influence 16 bit	Reserve 8 bit	Version 8 bit	Counter 63–48 bit	47–40 bit	39–32 bit	31–1 bit	0 bit
Waveguide Length	Cal Std ID	0x0500	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Waveguide Width	Cal Std ID	0x0501	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Waveguide Height	Cal Std ID	0x0502	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Waveguide Radius	Cal Std ID	0x0503	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Waveguide Width Offset	Cal Std ID	0x0504	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Waveguide Height Offset	Cal Std ID	0x0505	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Waveguide Connector FEM	Cal Std ID	0x0506	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Simple Line Length	Cal Std ID	0x0600	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Simple Line G	Cal Std ID	0x0601	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Simple Line C	Cal Std ID	0x0602	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
On-Wafer Length	Cal Std ID	0x0610	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
On-Wafer Width of Ground Conductor	Cal Std ID	0x0611	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
On-Wafer Width of Signal Conductor	Cal Std ID	0x0612	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
On-Wafer Gap Width	Cal Std ID	0x0613	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
On-Wafer Conductor Thickness	Cal Std ID	0x0614	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
On-Wafer Surface Roughness	Cal Std ID	0x0620	0x00	0x00	0x0000	0x00	0x00	0x00000000	0



Table 10: Uncertainty Input IDs cont.

Unc Contribution	Global ID 128 bit	Influence 16 bit	Reserve 8 bit	Version 8 bit	Counter 63–48 bit	47–40 bit	39–32 bit	31–1 bit	0 bit
Power Cal Std Reflection Coefficient	Cal Std ID	0x0700	0x00	0x01	Frequency (mHz)				RI
Power Cal Std Calibration Factor	Cal Std ID	0x0701	0x00	0x01	Frequency (mHz)				0
Power Measurement Exp Statistical	Journal ID	0x0708	0x00	0x00	Meas Count	0x00	0x00	Freq	0
Power Measurement Exp Systematic	Journal ID	0x0709	0x00	0x00	Exp Count	0x00	0x00	Freq	0
Power Measurement Type A	Random ID	0x070A	0x00	0x00	0x0000	0x00	0x00	Freq	0
	Journal ID	0x070A	0x00	0x01	Meas Count	0x00	0x00	Freq	0
Power Measurement Noise	Journal ID	0x0710	0x00	0x00	Meas Count	0x00	0x00	Freq	0
Power Measurement Zero Offset	Journal ID	0x0711	0x00	0x00	Zero Count	0x00	0x00	0x00000000	0
Power Measurement Zero Drift	Journal ID	0x0712	0x00	0x00	Meas Count	0x00	0x00	Freq	0
Power Cal Std Aging Freq Corr	Cal Std ID	0x0720	0x00	0x00	0x0000	0x00	0x00	0x00000000	0
Power Cal Std Aging Freq Uncorr	Cal Std ID	0x0721	0x00	0x01	Frequency (mHz)				0
Power Measurement Drift Freq Corr	Journal ID	0x0722	0x00	0x00	Meas Count	0x00	0x00	0x00000000	0
Power Measurement Drift Freq Uncorr	Journal ID	0x0723	0x00	0x00	Meas Count	0x00	0x00	Freq	0
Power Cal Std Linearity	Cal Std ID	0x0724	0x00	0x00	0x0000	0x00	0x00	$10^6(p + 1000)$	0
Power Cal Std Power Meter	Cal Std ID	0x0725	0x00	0x00	0x0000	0x00	0x00	$10^6(p + 1000)$	0
Power Measurement Repeatability	Journal ID	0x0726	0x00	0x00	Conn Count	Port	0x00	Freq	0
Power Measurement Temperature	Journal ID	0x0727	0x00	0x00	Meas Count	0x00	0x00	Freq	0
Unknown Mixer	Cal Std ID	0x0780	0x00	0x00	Frequency (mHz)				MP



## A S-Parameter Tools

### A.1 Cascading

Cascading of two S-parameter sets is described in [28] and [29].

#### A.1.1 Cascading of a 2N-Port and a N-Port

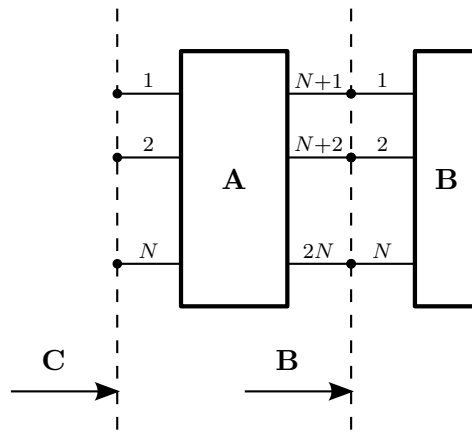


Figure 19: Cascading of a 2N-port (A) and a N-port (B)

One can use the following equation to cascade a 2N-port and a N-port. The result will be the N-port C

$$\mathbf{C} = \mathbf{A}_{00} + \mathbf{A}_{01} (\mathbf{I} - \mathbf{B}\mathbf{A}_{11})^{-1} \mathbf{B}\mathbf{A}_{10} \quad (303)$$

with

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{00} & \mathbf{A}_{01} \\ \mathbf{A}_{10} & \mathbf{A}_{11} \end{pmatrix}. \quad (304)$$

The variables  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{C}$  are S-parameter matrices.  $\mathbf{A}$  is a 2N-port and  $\mathbf{B}$  is a N-port. A simplified notation can be achieved by introducing a new operator

$$\mathbf{C} = \mathbf{A} \oplus \mathbf{B}. \quad (305)$$

To find the reverse function, equation 303 can be rewritten as

$$\mathbf{A}_{01}^{-1} (\mathbf{C} - \mathbf{A}_{00}) \mathbf{A}_{10}^{-1} = (\mathbf{I} - \mathbf{B}\mathbf{A}_{11})^{-1} \mathbf{B}. \quad (306)$$

Setting

$$\mathbf{X} = \mathbf{A}_{01}^{-1} (\mathbf{C} - \mathbf{A}_{00}) \mathbf{A}_{10}^{-1} \quad (307)$$

and rearranging equation 306 yields

$$\mathbf{B} = \mathbf{X} (\mathbf{I} + \mathbf{A}_{11}\mathbf{X})^{-1}. \quad (308)$$

Equations 307 and 308 can be used to de-cascade the 2N-port  $\mathbf{A}$  from N-port  $\mathbf{C}$  and to obtain the N-port  $\mathbf{B}$ . Similar to the notation in equation 305 a new operator can be introduced

$$\mathbf{B} = \mathbf{C} \ominus \mathbf{A}. \quad (309)$$

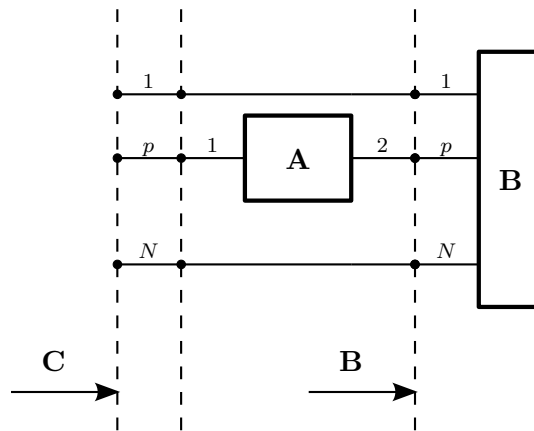


Figure 20: Cascading of a 2-port (A) and a  $N$ -port (B)

### A.1.2 Cascading of a 2-Port and a N-Port

Cascading of a 2-port and a  $N$ -port can be computed with

$$C_{ij} = \begin{cases} A_{11} + \frac{A_{21}B_{ij}A_{12}}{1-A_{22}B_{pp}} & , i = j = p \\ \frac{A_{21}B_{ij}}{1-A_{22}B_{pp}} & , i \neq p \wedge j = p \\ \frac{B_{ij}A_{12}}{1-A_{22}B_{pp}} & , i = p \wedge j \neq p \\ B_{ij} + \frac{B_{pj}A_{22}B_{ip}}{1-A_{22}B_{pp}} & , i \neq p \wedge j \neq p \end{cases} \quad (310)$$

Here the variables  $A$ ,  $B$  and  $C$  denote S-parameter matrices. The result  $C$  is a  $N$ -port with a 2-port cascaded to port  $p$  of the original  $N$ -port  $B$ .

### A.1.3 Cascading of a 2-Port and a 2-Port

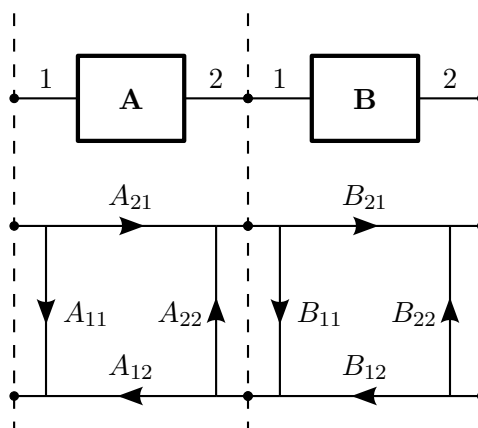


Figure 21: Cascading of a 2-port (A) and a 2-port (B)



## METAS VNA Tools - Math Reference V3.0.1

A special case of cascading is a 2-port and a 2-port. Equation 310 can be rewritten as

$$C_{11} = A_{11} + \frac{A_{21}B_{11}A_{12}}{1 - A_{22}B_{11}} \quad (311)$$

$$C_{21} = \frac{A_{21}B_{21}}{1 - A_{22}B_{11}} \quad (312)$$

$$C_{12} = \frac{B_{12}A_{12}}{1 - A_{22}B_{11}} \quad (313)$$

$$C_{22} = B_{22} + \frac{B_{12}A_{22}B_{21}}{1 - A_{22}B_{11}} \quad (314)$$

and a new operator can be introduced

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B}. \quad (315)$$

To invert a 2-port, equations 311 to 314 can be rewritten as

$$B_{11} = \frac{A_{11}}{A_{11}A_{22} - A_{21}A_{12}} \quad (316)$$

$$B_{21} = \frac{1 - A_{22}B_{11}}{A_{21}} \quad (317)$$

$$B_{12} = \frac{1 - A_{22}B_{11}}{A_{12}} \quad (318)$$

$$B_{22} = -\frac{B_{12}A_{22}B_{21}}{1 - A_{22}B_{11}} \quad (319)$$

with  $C_{11} = C_{22} = 0$  and  $C_{21} = C_{12} = 1$ . And a new operator for inverting a 2-port can be introduced.

$$\mathbf{B} = \mathbf{A}^{\ominus 1}. \quad (320)$$

### A.1.4 Cascading of a 2-Port and a 1-Port

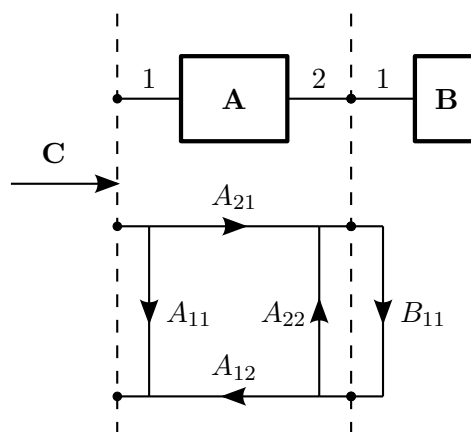


Figure 22: Cascading a 2-port (A) and a 1-port (B)

Another special case of cascading is a 2-port and a 1-port. Then equation 310 can be rewritten as

$$C_{11} = A_{11} + \frac{A_{21}B_{11}A_{12}}{1 - A_{22}B_{11}}. \quad (321)$$



### A.2 Three Port Tools

Three port tools are used to characterize the S-parameters of power splitters, power dividers and directional couplers.

#### A.2.1 Cascading of a 3-Port and a 1-Port

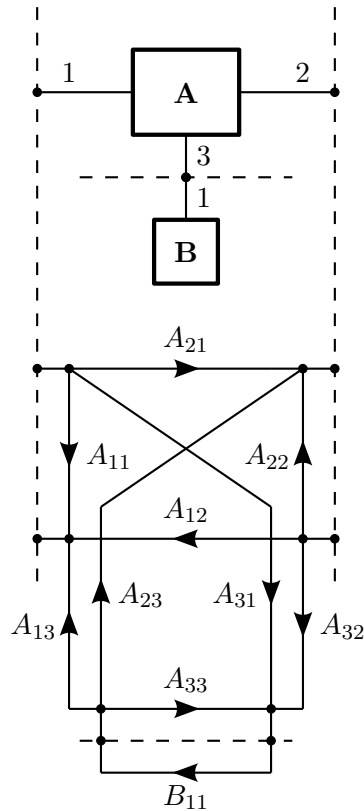


Figure 23: Cascading a 3-port (A) and a 1-port (B)

Another special case of cascading is a 3-port (A) and a 1-port (B) result in a 2-port (C). Then equation 310 can be rewritten as

$$C_{11} = A_{11} + \frac{A_{13}B_{11}A_{31}}{1 - A_{33}B_{11}} \quad (322)$$

$$C_{21} = A_{21} + \frac{A_{23}B_{11}A_{31}}{1 - A_{33}B_{11}} \quad (323)$$

$$C_{12} = A_{12} + \frac{A_{13}B_{11}A_{32}}{1 - A_{33}B_{11}} \quad (324)$$

$$C_{22} = A_{22} + \frac{A_{23}B_{11}A_{32}}{1 - A_{33}B_{11}} \quad (325)$$



### A.2.2 Compute a 3-Port using a linear equation system

One can compute the 3-port  $\mathbf{A}$  by using at least three 2-port measurements  $\mathbf{C}^{(i)}$  of the cascaded 3-port  $\mathbf{A}$  with different 1-port standards  $\mathbf{B}^{(i)}$ . This can be used to characterize the S-parameters of a reciprocal 3-port, e.g.: a power splitter, using a 2-port VNA. Equations 322 to 325 can be rewritten as

$$C_{11} = A_{11} + (A_{13}A_{31} - A_{11}A_{33}) B_{11} + A_{33}B_{11}C_{11} \quad (326)$$

$$C_{21} = A_{21} + (A_{23}A_{31} - A_{21}A_{33}) B_{11} + A_{33}B_{11}C_{21} \quad (327)$$

$$C_{12} = A_{12} + (A_{13}A_{32} - A_{12}A_{33}) B_{11} + A_{33}B_{11}C_{12} \quad (328)$$

$$C_{22} = A_{22} + (A_{23}A_{32} - A_{22}A_{33}) B_{11} + A_{33}B_{11}C_{22}. \quad (329)$$

In a next step one can set up the following linear equation system

$$\begin{pmatrix} 1 & 0 & 0 & 0 & B_{11}^{(1)} & 0 & 0 & 0 & B_{11}^{(1)}C_{11}^{(1)} \\ 0 & 1 & 0 & 0 & 0 & B_{11}^{(1)} & 0 & 0 & B_{11}^{(1)}C_{21}^{(1)} \\ 0 & 0 & 1 & 0 & 0 & 0 & B_{11}^{(1)} & 0 & B_{11}^{(1)}C_{12}^{(1)} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & B_{11}^{(1)} & B_{11}^{(1)}C_{22}^{(1)} \\ \hline 1 & 0 & 0 & 0 & B_{11}^{(2)} & 0 & 0 & 0 & B_{11}^{(2)}C_{11}^{(2)} \\ 0 & 1 & 0 & 0 & 0 & B_{11}^{(2)} & 0 & 0 & B_{11}^{(2)}C_{21}^{(2)} \\ 0 & 0 & 1 & 0 & 0 & 0 & B_{11}^{(2)} & 0 & B_{11}^{(2)}C_{12}^{(2)} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & B_{11}^{(2)} & B_{11}^{(2)}C_{22}^{(2)} \\ \hline 1 & 0 & 0 & 0 & B_{11}^{(3)} & 0 & 0 & 0 & B_{11}^{(3)}C_{11}^{(3)} \\ 0 & 1 & 0 & 0 & 0 & B_{11}^{(3)} & 0 & 0 & B_{11}^{(3)}C_{21}^{(3)} \\ 0 & 0 & 1 & 0 & 0 & 0 & B_{11}^{(3)} & 0 & B_{11}^{(3)}C_{12}^{(3)} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & B_{11}^{(3)} & B_{11}^{(3)}C_{22}^{(3)} \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{21} \\ A_{12} \\ A_{22} \\ T_{11} \\ T_{21} \\ T_{12} \\ T_{22} \\ A_{33} \end{pmatrix} = \begin{pmatrix} C_{11}^{(1)} \\ C_{21}^{(1)} \\ C_{12}^{(1)} \\ C_{22}^{(1)} \\ \hline C_{11}^{(2)} \\ C_{21}^{(2)} \\ C_{12}^{(2)} \\ C_{22}^{(2)} \\ \hline C_{11}^{(3)} \\ C_{21}^{(3)} \\ C_{12}^{(3)} \\ C_{22}^{(3)} \end{pmatrix} \quad (330)$$

with

$$T_{11} = A_{13}A_{31} - A_{11}A_{33} \quad (331)$$

$$T_{21} = A_{23}A_{31} - A_{21}A_{33} \quad (332)$$

$$T_{12} = A_{13}A_{32} - A_{12}A_{33} \quad (333)$$

$$T_{22} = A_{23}A_{32} - A_{22}A_{33}. \quad (334)$$

One can compute the missing transmission parameters of the 3-port  $\mathbf{A}$  using the following equations

$$A_{31} = A_{13} = \sqrt{T_{11} + A_{11}A_{33}} \quad (335)$$

and

$$A_{32} = A_{23} = \sqrt{T_{22} + A_{22}A_{33}}. \quad (336)$$

### A.2.3 Compute a 3-Port using a non-linear optimization algorithm

To find the S-parameters with the smallest uncertainties of the reciprocal 3-port  $\mathbf{A}$  a non-linear optimization algorithm with covariance weighting is used. One can set up the following



objective function

$$\begin{pmatrix}
 A_{11} + \frac{A_{13}B_{11}^{(1)}A_{31}}{1-A_{33}B_{11}^{(1)}} - C_{11}^{(1)} \\
 A_{21} + \frac{A_{23}B_{11}^{(1)}A_{31}}{1-A_{33}B_{11}^{(1)}} - C_{21}^{(1)} \\
 A_{12} + \frac{A_{13}B_{11}^{(1)}A_{32}}{1-A_{33}B_{11}^{(1)}} - C_{12}^{(1)} \\
 A_{22} + \frac{A_{23}B_{11}^{(1)}A_{32}}{1-A_{33}B_{11}^{(1)}} - C_{22}^{(1)} \\
 \hline
 A_{11} + \frac{A_{13}B_{11}^{(2)}A_{31}}{1-A_{33}B_{11}^{(2)}} - C_{11}^{(2)} \\
 A_{21} + \frac{A_{23}B_{11}^{(2)}A_{31}}{1-A_{33}B_{11}^{(2)}} - C_{21}^{(2)} \\
 A_{12} + \frac{A_{13}B_{11}^{(2)}A_{32}}{1-A_{33}B_{11}^{(2)}} - C_{12}^{(2)} \\
 A_{22} + \frac{A_{23}B_{11}^{(2)}A_{32}}{1-A_{33}B_{11}^{(2)}} - C_{22}^{(2)} \\
 \hline
 A_{11} + \frac{A_{13}B_{11}^{(3)}A_{31}}{1-A_{33}B_{11}^{(3)}} - C_{11}^{(3)} \\
 A_{21} + \frac{A_{23}B_{11}^{(3)}A_{31}}{1-A_{33}B_{11}^{(3)}} - C_{21}^{(3)} \\
 A_{12} + \frac{A_{13}B_{11}^{(3)}A_{32}}{1-A_{33}B_{11}^{(3)}} - C_{12}^{(3)} \\
 A_{22} + \frac{A_{23}B_{11}^{(3)}A_{32}}{1-A_{33}B_{11}^{(3)}} - C_{22}^{(3)}
 \end{pmatrix} \quad (337)$$

where  $A_{21} = A_{12}$ ,  $A_{31} = A_{13}$  and  $A_{32} = A_{23}$ .

The start values for the 3-port  $\mathbf{A}$  are computed using the linear equation system, see section A.2.2.

### A.2.4 Equivalent Source Match of a Power Splitter

The following equations describe the equivalent source match for port 2 of a power splitter

$$\Gamma_{Eq2} = S_{22} - \frac{S_{32}S_{21}}{S_{31}} \quad (338)$$

and for port 3 of a power splitter

$$\Gamma_{Eq3} = S_{33} - \frac{S_{23}S_{31}}{S_{21}}. \quad (339)$$

The ratio  $S_{21}/S_{31}$  describes the tracking of a power splitter.

### A.2.5 Directivity of a Directional Coupler

The following equation describes the directivity of a directional coupler

$$D = \left| \frac{S_{32}}{S_{31}} \right| \quad (340)$$

where  $S_{31}$  is the transmission from port 1 to port 3 (coupling) and  $S_{32}$  is the transmission from port 2 to port 3 (isolation).



### A.3 Transmission Line Junction

A transmission line junction 2-port can be used to change the reference impedance. The transformation of the reference impedance is described in [30]. VNA Tools is using pseudo waves for changing the reference impedance, see section A.3.2.

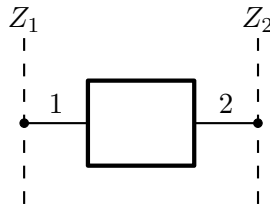


Figure 24: Transmission line junction

#### A.3.1 Power Waves

The S-parameters of a transmission line junction using power waves are given by the following equations

$$S_{11} = \frac{(Z_2 - Z_1)}{(Z_1 + Z_2)} \quad (341)$$

$$S_{21} = \frac{2\sqrt{Z_1 Z_2}}{(Z_1 + Z_2)} \quad (342)$$

$$S_{12} = \frac{2\sqrt{Z_1 Z_2}}{(Z_1 + Z_2)} \quad (343)$$

$$S_{22} = \frac{(Z_1 - Z_2)}{(Z_1 + Z_2)}. \quad (344)$$

The reference impedances of these S-parameters are  $Z_1$  for port one and  $Z_2$  for port two.

#### A.3.2 Pseudo Waves

The S-parameters of a transmission line junction using pseudo waves are given by the following equations

$$S_{11} = \frac{(Z_2 - Z_1)}{(Z_1 + Z_2)} \quad (345)$$

$$S_{21} = \frac{2Z_2 \left| \frac{Z_1}{Z_2} \right| \sqrt{\frac{\text{Re}(Z_2)}{\text{Re}(Z_1)}}}{(Z_1 + Z_2)} \quad (346)$$

$$S_{12} = \frac{2Z_1 \left| \frac{Z_2}{Z_1} \right| \sqrt{\frac{\text{Re}(Z_1)}{\text{Re}(Z_2)}}}{(Z_1 + Z_2)} \quad (347)$$

$$S_{22} = \frac{(Z_1 - Z_2)}{(Z_1 + Z_2)}. \quad (348)$$

The reference impedances of these S-parameters are  $Z_1$  for port one and  $Z_2$  for port two.



#### A.4 Transmission Line Section

The S-parameters of a transmission line section are given by the following equations

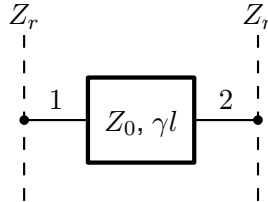


Figure 25: Transmission line section

$$S_{11} = \frac{(Z_0^2 - Z_r^2) \sinh \gamma l}{2Z_0 Z_r \cosh \gamma l + (Z_0^2 + Z_r^2) \sinh \gamma l} \quad (349)$$

$$S_{21} = \frac{2Z_0 Z_r}{2Z_0 Z_r \cosh \gamma l + (Z_0^2 + Z_r^2) \sinh \gamma l} \quad (350)$$

$$S_{12} = \frac{2Z_0 Z_r}{2Z_0 Z_r \cosh \gamma l + (Z_0^2 + Z_r^2) \sinh \gamma l} \quad (351)$$

$$S_{22} = \frac{(Z_0^2 - Z_r^2) \sinh \gamma l}{2Z_0 Z_r \cosh \gamma l + (Z_0^2 + Z_r^2) \sinh \gamma l} \quad (352)$$

Where  $Z_0$  is the characteristic impedance and  $\gamma l$  is the propagation constant times the length. The reference impedance at both ports is  $Z_r$ .



### A.4.1 Lossy Coaxial Transmission Line Section

A lossy coaxial transmission line section is described in [31].

$$\sigma = \sigma_{DC} - \sigma_{HF} \sqrt{\frac{f}{1 \text{ GHz}}} \quad (353)$$

$$k = \omega \sqrt{\mu \epsilon} \quad (354)$$

$$d_0 = \frac{\sqrt{\frac{2}{\sigma \omega \mu}} \left(1 + \frac{b}{a}\right)}{4b \ln\left(\frac{b}{a}\right)} \quad (355)$$

$$F_0 = \frac{\frac{b^2}{a^2} - 1}{2 \ln\left(\frac{b}{a}\right)} - \frac{\frac{b}{a} \ln\left(\frac{b}{a}\right)}{\frac{b}{a} + 1} - \frac{1}{2} \left(\frac{b}{a} + 1\right) \quad (356)$$

$$C'_0 = \frac{2\pi \epsilon}{\ln\left(\frac{b}{a}\right)} \quad (357)$$

$$L'_0 = \frac{\mu \ln\left(\frac{b}{a}\right)}{2\pi} \quad (358)$$

$$R' = 2\omega L'_0 d_0 \left(1 - \frac{k^2 a^2 F_0}{2}\right) \quad (359)$$

$$L' = L'_0 \left(1 + 2d_0 \left(1 - \frac{k^2 a^2 F_0}{2}\right)\right) \quad (360)$$

$$G' = \omega C'_0 d_0 k^2 a^2 F_0 \quad (361)$$

$$C' = C'_0 \left(1 + d_0 k^2 a^2 F_0\right) \quad (362)$$

$$Z' = R' + j\omega L' \quad (363)$$

$$Y' = G' + j\omega C' \quad (364)$$

$$\gamma = \sqrt{Z' Y'} \quad (365)$$

$$Z_0 = \sqrt{\frac{Z'}{Y'}} \quad (366)$$

Where  $\mu$  and  $\epsilon$  are permeability and permittivity of the dielectric. The conductors are characterized by their conductivity  $\sigma_{DC}$  and  $\sigma_{HF}$ . The frequency is  $f$  and the angular frequency is  $\omega$ . Outer conductor inner radius  $b$  and inner conductor outer radius  $a$  describe the geometry of the line.



## B Wave Parameter Tools

The S-parameter matrix  $S$  is defined with the following equation

$$\mathbf{b} = \mathbf{S}\mathbf{a} \tag{367}$$

or using explicit components

$$\begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} = \begin{pmatrix} S_{1,1} & \dots & S_{1,N} \\ \vdots & \ddots & \vdots \\ S_{N,1} & \dots & S_{N,N} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \tag{368}$$

where  $\mathbf{b}$  is a column vector for the test receiver wave quantities and  $\mathbf{a}$  is a column vector for the reference receiver wave quantities.  $N$  is the number of ports.

The above equation can be generalized by adding additional columns to  $\mathbf{b}$ ,  $\mathbf{a}$ . This yields into the following equation

$$\mathbf{B} = \mathbf{S}\mathbf{A} \tag{369}$$

or using explicit components

$$\begin{pmatrix} b_{1,1} & \dots & b_{1,N} \\ \vdots & \ddots & \vdots \\ b_{N,1} & \dots & b_{N,N} \end{pmatrix} = \begin{pmatrix} S_{1,1} & \dots & S_{1,N} \\ \vdots & \ddots & \vdots \\ S_{N,1} & \dots & S_{N,N} \end{pmatrix} \begin{pmatrix} a_{1,1} & \dots & a_{1,N} \\ \vdots & \ddots & \vdots \\ a_{N,1} & \dots & a_{N,N} \end{pmatrix} \tag{370}$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are wave parameter matrices. The row index indicates the port and the column index indicates the stimulus port.

### B.1 Cascading

Figure 26 describes cascading of an S-parameter matrix  $\mathbf{Y}$ , which consist of a product of wave parameter matrices, with a conventional S-parameter matrix  $\mathbf{X}$ . The variables  $\mathbf{X}$ ,  $\mathbf{Y}$  and  $\mathbf{Z}$  are S-parameter matrices. The variables  $\mathbf{B}_Y$ ,  $\mathbf{A}_Y$  and  $\mathbf{B}_Z$ ,  $\mathbf{A}_Z$  are wave parameter matrices.  $\mathbf{X}$  is a  $2N$ -port and  $\mathbf{Y}$  is a  $N$ -port.  $\mathbf{Z}$  will be a  $N$ -port. According to [28] and [29] the S-parameter matrix  $\mathbf{X}$  can be split up into

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_{00} & \mathbf{X}_{01} \\ \mathbf{X}_{10} & \mathbf{X}_{11} \end{pmatrix} \tag{371}$$

and the following set of equations holds

$$\mathbf{B}_Z = \mathbf{X}_{00}\mathbf{A}_Z + \mathbf{X}_{01}\mathbf{B}_Y \tag{372}$$

$$\mathbf{A}_Y = \mathbf{X}_{10}\mathbf{A}_Z + \mathbf{X}_{11}\mathbf{B}_Y. \tag{373}$$

One can solve equation 373 for  $\mathbf{A}_Z$

$$\mathbf{A}_Z = \mathbf{X}_{10}^{-1}\mathbf{A}_Y - \mathbf{X}_{10}^{-1}\mathbf{X}_{11}\mathbf{B}_Y \tag{374}$$

and plug the result into 372

$$\mathbf{B}_Z = \mathbf{X}_{00}\mathbf{X}_{10}^{-1}\mathbf{A}_Y - (\mathbf{X}_{00}\mathbf{X}_{10}^{-1}\mathbf{X}_{11} - \mathbf{X}_{01})\mathbf{B}_Y. \tag{375}$$

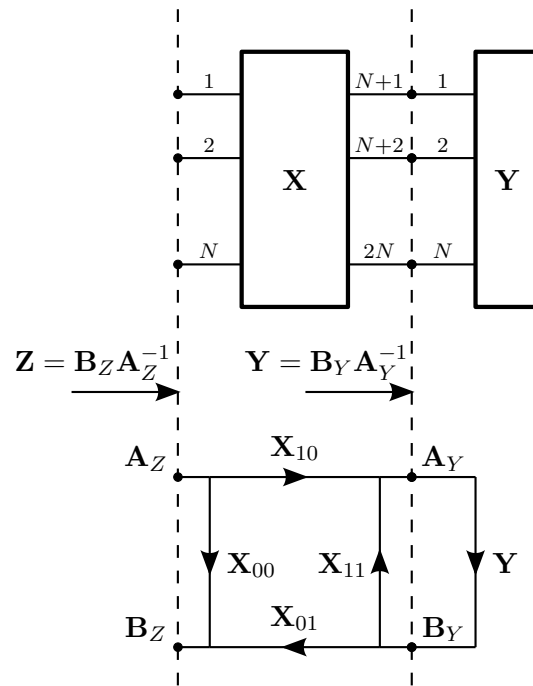


Figure 26: Cascading of a  $2N$ -port S-parameter ( $\mathbf{X}$ ) and the  $N$ -port wave parameter matrices ( $\mathbf{B}_Y, \mathbf{A}_Y$ )

The above two equations 374 and 375 can be used to cascade a  $2N$ -port S-parameter and the  $N$ -port wave parameter matrices. The result will be the  $N$ -port wave parameter matrices  $\mathbf{B}_Z, \mathbf{A}_Z$ .

A simplified notation can be achieved by introducing a new operator

$$(\mathbf{B}_Z, \mathbf{A}_Z) = \mathbf{X} \odot (\mathbf{B}_Y, \mathbf{A}_Y). \quad (376)$$

## B.2 De-cascading

One can solve equation 372 for  $\mathbf{B}_Y$

$$\mathbf{B}_Y = \mathbf{X}_{01}^{-1} \mathbf{B}_Z - \mathbf{X}_{01}^{-1} \mathbf{X}_{00} \mathbf{A}_Z \quad (377)$$

and plug the result into 373

$$\mathbf{A}_Y = \mathbf{X}_{11} \mathbf{X}_{01}^{-1} \mathbf{B}_Z - (\mathbf{X}_{11} \mathbf{X}_{01}^{-1} \mathbf{X}_{00} - \mathbf{X}_{10}) \mathbf{A}_Z. \quad (378)$$

The above two equations 377 and 378 can be used to de-cascade a  $2N$ -port S-parameter  $\mathbf{X}$  from the  $N$ -port wave parameter matrices  $\mathbf{B}_Z, \mathbf{A}_Z$ . The result will be the  $N$ -port wave parameter matrices  $\mathbf{B}_Y, \mathbf{A}_Y$ .

A simplified notation can be achieved by introducing a new operator

$$(\mathbf{B}_Y, \mathbf{A}_Y) = (\mathbf{B}_Z, \mathbf{A}_Z) \oslash \mathbf{X}. \quad (379)$$



### B.3 SOLT Error Correction

Figure 27 describes the SOLT error correction of a 2-port using error terms for each switch position  $x$  of the source.

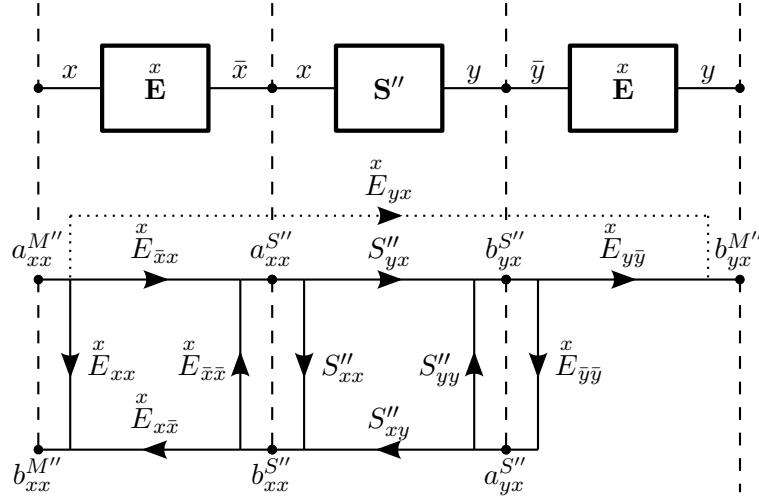


Figure 27: SOLT Error Correction

The following set of equations holds for the VNA test receivers wave quantities

$$b_{xx}^{M''} = a_{xx}^{M''} E_{xx}^x + b_{xx}^{S''} E_{x\bar{x}}^x \quad (380)$$

$$b_{yx}^{M''} = a_{xx}^{M''} E_{yx}^x + b_{yx}^{S''} E_{y\bar{y}}^x \quad (381)$$

where  $\bar{x} = N + x$ ,  $\bar{y} = N + y$  and  $N$  is the number of ports. The above equations can be solved for the reflected and transmitted wave quantities of the device under test

$$b_{xx}^{S''} = (b_{xx}^{M''} - a_{xx}^{M''} E_{xx}^x) / E_{x\bar{x}}^x \quad (382)$$

$$b_{yx}^{S''} = (b_{yx}^{M''} - a_{xx}^{M''} E_{yx}^x) / E_{y\bar{y}}^x. \quad (383)$$

The following set of equations holds for the incident wave quantities of the device under test

$$a_{xx}^{S''} = b_{xx}^{S''} E_{\bar{x}\bar{x}}^x + a_{xx}^{M''} E_{\bar{x}x}^x \quad (384)$$

$$a_{yx}^{S''} = b_{yx}^{S''} E_{\bar{y}\bar{y}}^x. \quad (385)$$

This concept can be generalized to  $N$ -ports by adding additional receiving ports.



### B.4 Switch Terms

The S-parameter matrix  $\mathbf{S}$  is defined with the following equation

$$\mathbf{B} = \mathbf{S}\mathbf{A} \quad (386)$$

or using explicit components

$$\begin{pmatrix} b_{1,1} & \dots & b_{1,N} \\ \vdots & \ddots & \vdots \\ b_{N,1} & \dots & b_{N,N} \end{pmatrix} = \begin{pmatrix} S_{1,1} & \dots & S_{1,N} \\ \vdots & \ddots & \vdots \\ S_{N,1} & \dots & S_{N,N} \end{pmatrix} \begin{pmatrix} a_{1,1} & \dots & a_{1,N} \\ \vdots & \ddots & \vdots \\ a_{N,1} & \dots & a_{N,N} \end{pmatrix} \quad (387)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are wave parameter matrices. The row index indicates the port and the column index indicates the stimulus port.  $N$  is the number of ports.

In most of the cases only the diagonal of the wave parameter matrix  $\mathbf{A}$  is measured, because some VNA manufactures can not collect all test and reference receivers in the same sweep. Another reason is that measuring the complete matrix  $\mathbf{A}$  will introduce more noise to the S-parameters  $\mathbf{S}$  because the measured signal at the passive reference receivers is closer to the noise floor. This yields into the following equation

$$\mathbf{B} = \mathbf{S} \underbrace{\mathbf{C}\mathbf{A}}_{\mathbf{A}^{diag}} \quad (388)$$

The off-diagonal of the matrix  $\mathbf{A}$  can be reconstructed with the correction matrix  $\mathbf{C}$

$$\mathbf{C} = \mathbf{A}\mathbf{A}_{diag}^{-1} \quad (389)$$

or using explicit components

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{a_{1,2}}{a_{1,1}} & \frac{a_{1,3}}{a_{1,1}} & \dots & \frac{a_{1,N}}{a_{1,1}} \\ \frac{a_{2,1}}{a_{2,2}} & 1 & \frac{a_{2,3}}{a_{2,2}} & \dots & \frac{a_{2,N}}{a_{2,2}} \\ \frac{a_{3,1}}{a_{3,3}} & \frac{a_{3,2}}{a_{3,3}} & 1 & \dots & \frac{a_{3,N}}{a_{3,3}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{a_{N,1}}{a_{N,N}} & \frac{a_{N,2}}{a_{N,N}} & \frac{a_{N,3}}{a_{N,N}} & \dots & 1 \end{pmatrix} \quad (390)$$

One can expand the off-diagonal elements of the matrix  $\mathbf{C}$  with  $\frac{b_{i,j}}{b_{i,i}}$

$$\mathbf{C} = \begin{pmatrix} 1 & \frac{b_{1,2}}{a_{2,2}} \frac{a_{1,2}}{b_{1,2}} & \frac{b_{1,3}}{a_{3,3}} \frac{a_{1,3}}{b_{1,3}} & \dots & \frac{b_{1,N}}{a_{N,N}} \frac{a_{1,N}}{b_{1,N}} \\ \frac{b_{2,1}}{a_{1,1}} \frac{a_{2,1}}{b_{2,1}} & 1 & \frac{b_{2,3}}{a_{3,3}} \frac{a_{2,3}}{b_{2,3}} & \dots & \frac{b_{2,N}}{a_{N,N}} \frac{a_{2,N}}{b_{2,N}} \\ \frac{b_{3,1}}{a_{1,1}} \frac{a_{3,1}}{b_{3,1}} & \frac{b_{3,2}}{a_{2,2}} \frac{a_{3,2}}{b_{3,2}} & 1 & \dots & \frac{b_{3,N}}{a_{N,N}} \frac{a_{3,N}}{b_{3,N}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b_{N,1}}{a_{1,1}} \frac{a_{N,1}}{b_{N,1}} & \frac{b_{N,2}}{a_{2,2}} \frac{a_{N,2}}{b_{N,2}} & \frac{b_{N,3}}{a_{3,3}} \frac{a_{N,3}}{b_{N,3}} & \dots & 1 \end{pmatrix} \quad (391)$$

It can be shown that the ratio  $\frac{a_{i,j}}{b_{i,j}}$  is independent of the switch position  $j$  of the source. This ratio is called switch term of port  $i$ . The switch terms can be measured once. Afterwards they can be used to reconstruct the full matrix  $\mathbf{A}$ .



### B.5 Absorbed Power

The absorbed power by a device under test is described in [30]. VNA Tools is using pseudo waves to compute the absorbed power, see section B.5.4.

#### B.5.1 Power Waves 1-Port

The following equation describes the absorbed power of a 1-port using power waves

$$P = |a|^2 - |b|^2 \quad (392)$$

where  $a$  is incident wave and  $b$  is the reflected wave. The above equations can be rewritten as

$$P = |a|^2 (1 - |S|^2) \quad (393)$$

where  $S$  is the reflection coefficient of the 1-port.

#### B.5.2 Pseudo Waves 1-Port

The following equation describes the absorbed power of a 1-port using pseudo waves

$$P = |a|^2 - |b|^2 + 2 \operatorname{Im}(ab^*) \frac{\operatorname{Im}(Z_r)}{\operatorname{Re}(Z_r)} \quad (394)$$

where  $a$  is incident wave and  $b$  is the reflected wave. The operator  $*$  denotes the conjugate transpose and  $Z_r$  is the complex reference impedance. The above equation can be rewritten as

$$P = |a|^2 - |b|^2 - 2 \operatorname{Im} \left( aa^* \frac{b}{a} \right) \frac{\operatorname{Im}(Z_r)}{\operatorname{Re}(Z_r)} \quad (395)$$

which yields to

$$P = |a|^2 \left( 1 - |S|^2 - 2 \operatorname{Im}(S) \frac{\operatorname{Im}(Z_r)}{\operatorname{Re}(Z_r)} \right) \quad (396)$$

where  $S$  is the reflection coefficient of the 1-port.

In the case where the imaginary part of  $Z_r$  is zero, the absorbed power using power waves and the absorbed power using pseudo waves are the same.

#### B.5.3 Power Waves N-Port

The following equation describes the absorbed power of a  $N$ -port using power waves

$$P_k = \sum_{j=1}^N |a_{j,k}|^2 \left( 1 - \sum_{i=1}^N |S_{i,j}|^2 \right). \quad (397)$$

The absorbed power can be computed for each source position  $k$ .

#### B.5.4 Pseudo Waves N-Port

In a first step the complex reference impedance of  $N$ -port is changed to a real reference impedance by cascading a transmission line junction to each port, see sections A.3.2 and B.1. In a second step the absorbed power is computed by using equation 397.



## C Mixed-Mode S-Parameters

The mixed-mode S-parameters are described in [32] and [33].

### C.1 Single-Ended S-Parameters to Mixed-Mode S-Parameters

The single-ended S-parameter matrix

$$\mathbf{S} = \begin{pmatrix} S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} \\ S_{2,1} & S_{2,2} & S_{2,3} & S_{2,4} \\ S_{3,1} & S_{2,3} & S_{3,3} & S_{3,4} \\ S_{4,1} & S_{2,4} & S_{4,3} & S_{4,4} \end{pmatrix} \quad (398)$$

and the mode conversion matrix

$$\mathbf{M} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad (399)$$

and the relation for the mixed-mode matrix

$$\mathbf{S}_M = \mathbf{M}\mathbf{S}\mathbf{M}^{-1} \quad (400)$$

yield

$$\mathbf{S}_M = \begin{pmatrix} \frac{S_{1,1}-S_{1,2}-S_{2,1}+S_{2,2}}{2} & \frac{S_{1,1}+S_{1,2}-S_{2,1}-S_{2,2}}{2} & \frac{S_{1,3}-S_{1,4}-S_{2,3}+S_{2,4}}{2} & \frac{S_{1,3}+S_{1,4}-S_{2,3}-S_{2,4}}{2} \\ \frac{S_{1,1}-S_{1,2}+S_{2,1}-S_{2,2}}{2} & \frac{S_{1,1}+S_{1,2}+S_{2,1}+S_{2,2}}{2} & \frac{S_{1,3}-S_{1,4}+S_{2,3}-S_{2,4}}{2} & \frac{S_{1,3}+S_{1,4}+S_{2,3}+S_{2,4}}{2} \\ \frac{S_{3,1}-S_{3,2}-S_{4,1}+S_{4,2}}{2} & \frac{S_{3,1}+S_{3,2}-S_{4,1}-S_{4,2}}{2} & \frac{S_{3,3}-S_{3,4}-S_{4,3}+S_{4,4}}{2} & \frac{S_{3,3}+S_{3,4}-S_{4,3}-S_{4,4}}{2} \\ \frac{S_{3,1}-S_{3,2}+S_{4,1}-S_{4,2}}{2} & \frac{S_{3,1}+S_{3,2}+S_{4,1}+S_{4,2}}{2} & \frac{S_{3,3}-S_{3,4}+S_{4,3}-S_{4,4}}{2} & \frac{S_{3,3}+S_{3,4}+S_{4,3}+S_{4,4}}{2} \end{pmatrix}. \quad (401)$$

The mode conversion matrix  $\mathbf{M}$  delivers the mixed-mode in the order 1d, 1c, 2d, 2c. The numbers refer to the logical port number where d is differential and c is common mode. The following matrix describes the mixed-mode S-parameters

$$\mathbf{S}_M = \begin{pmatrix} S_{1d,1d} & S_{1d,1c} & S_{1d,2d} & S_{1d,2c} \\ S_{1c,1d} & S_{1c,1c} & S_{1c,2d} & S_{1c,2c} \\ S_{2d,1d} & S_{2d,1c} & S_{2d,2d} & S_{2d,2c} \\ S_{2c,1d} & S_{2c,1c} & S_{2c,2d} & S_{2c,2c} \end{pmatrix}. \quad (402)$$

### C.2 Mixed-Mode S-Parameters to Single-Ended S-Parameters

One can solve equation 400 for the single-ended matrix  $\mathbf{S}$

$$\mathbf{S} = \mathbf{M}^{-1}\mathbf{S}_M\mathbf{M}. \quad (403)$$



## D Time Domain

The here used transformation of frequency domain S-parameters to time domain and time gating is described in [34].

### D.1 Frequency Domain to Time Domain

The transformation from frequency to time domain is shown in figure 28.

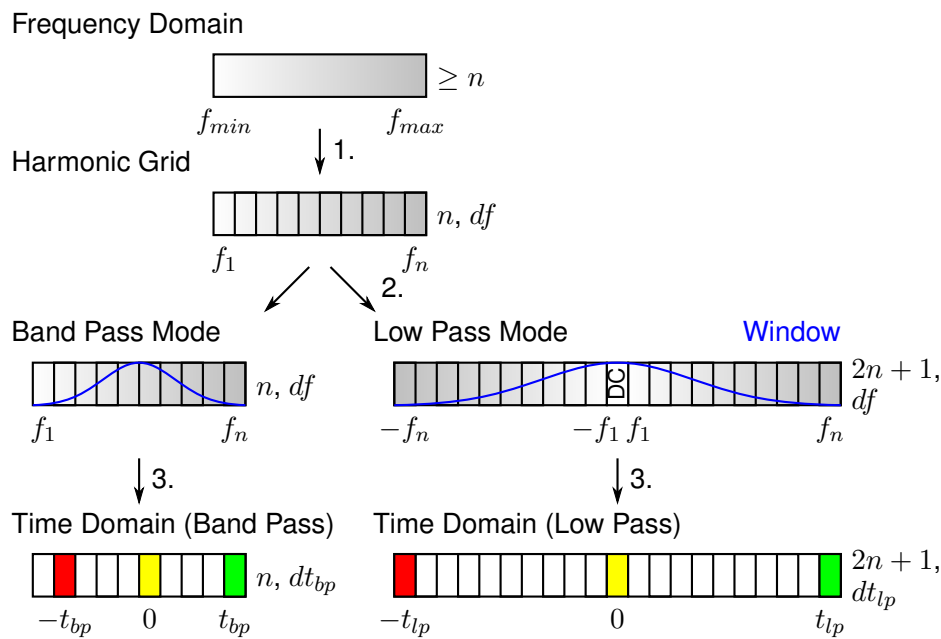


Figure 28: Illustration of the transformation from frequency domain to time domain using the band pass mode or the low pass mode.

1. The original data in frequency domain is interpolated on a harmonic grid. A harmonic grid is formed by a set of equidistant frequency points  $f_i$  ( $i = 1 \dots n$ ) with spacing  $df$ . The frequency step  $df$  of the harmonic grid is equal to the largest frequency step of the original data in frequency domain. The first frequency  $f_1$  of the harmonic grid has to be a multiple of  $df$  and  $f_{min} \leq f_1$ . The last frequency  $f_n = f_{max}$  of the harmonic grid has to be a multiple of  $df$ .
2. In the low pass mode, the data of the harmonic grid is mirrored (conjugate complex) to the negative frequencies and the DC point is added.
3. The data on the harmonic grid is multiplied with the window function and then transformed to time domain using the inverse DFT (discrete fourier transform).



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### D.1.1 Band Pass Mode

The band pass mode simulates a narrow-band TDR (Time Domain Reflectometer). It allows the user to identify locations of mismatches but does not indicate whether the mismatches are capacitive, inductive or resistive. However, it's suitable for displaying a magnitude proportional to the response of a TDR. Since the band pass mode does not include a DC value and low frequency values, only the impulse excitation is supported.

The time resolution is computed with the following equation.

$$dt_{bp} = \frac{1}{n \cdot df} \quad (404)$$

Note that the time resolution gets finer with a larger frequency span  $f_n - f_1$ .

The maximum time is computed with the following equation.

$$t_{bp} = \left\lfloor \frac{n-1}{2} \right\rfloor \cdot dt_{bp} \quad (405)$$

The operation  $\lfloor \ ]$  denotes rounding to the lower integer number. Note that the maximum time gets larger with a smaller frequency step  $df$ .

The minimum time is computed with the following equation.

$$-t'_{bp} = - \left\lceil \frac{n}{2} \right\rceil \cdot dt_{bp} \quad (406)$$

Note that for an odd number of points  $n$ , it is  $t'_{bp} = t_{bp}$ .

### D.1.2 Low Pass Mode

The low pass mode is used to simulate a traditional TDR measurement. This mode gives the user information to determine the type of discontinuity (R, L, or C) that is present. Low pass mode provides the best resolution (fastest rise time), and it may be used to either compute the step or impulse response of a device.

The low pass mode is less general than the band pass mode in that it places strict limitations on the frequency range of the measurement. The first frequency  $f_1$  must be equal to the frequency step  $df$ . The DC frequency response is either measured or extrapolated from the two lowest frequency points in frequency domain. The requirement for a DC point is the same limitation that exists for traditional TDR measurements.

The time resolution of the low pass mode is computed with the following equation.

$$dt_{lp} = \frac{1}{(2n+1) \cdot df} \quad (407)$$

Note that the time resolution of the low pass mode is about twice as fine as the time resolution of the band pass mode.

The maximum and minimum time is computed with the following equation.

$$t_{lp} = n \cdot dt_{lp} \quad (408)$$



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### D.1.3 Frequency and Time Uncertainty

In the following example the band pass mode is used. Let's assume a frequency uncertainty of

$$u(f_{stab}) = 10^{-6} \frac{\text{Hz}}{\text{Hz}} \quad (409)$$

and a frequency step  $df = 100 \text{ MHz}$  and  $n = 500$  measurement points. The time resolution would be computed with

$$dt_{bp} = \frac{1}{n \cdot df} = 20 \text{ ps} \pm 20 \text{ as}. \quad (410)$$

The maximum time would be

$$t_{bp} = \left[ \frac{n-1}{2} \right] \cdot dt_{bp} = 4.98 \text{ ns} \pm 4.98 \text{ fs} \quad (411)$$

and the minimum time would be

$$-t'_{bp} = - \left[ \frac{n}{2} \right] \cdot dt_{bp} = -5.00 \text{ ns} \pm 5.00 \text{ fs}. \quad (412)$$

The ratio of the worst case time uncertainty to the time resolution is

$$\frac{u(t_{bp} + t'_{bp})}{dt_{bp}} = \frac{9.98 \text{ fs}}{20 \text{ ps}} = (n-1) u(f_{stab}) = (500-1) 10^{-6} \frac{\text{Hz}}{\text{Hz}} = 4.99 \cdot 10^{-4}. \quad (413)$$

Note that the maximum and minimum time get larger the more measurement points  $n$  are used. Therefore the above ratio gets as well larger (worse) when more points  $n$  are used. The uncertainty of frequency and time is not taken into account by VNA Tools.



## D.2 Time Gating

This section describes the transformation from frequency to time domain, gating the data in time domain and transforming back to frequency domain.

### D.2.1 Band Pass Mode

For time gating, using the band pass mode, see figure 29.

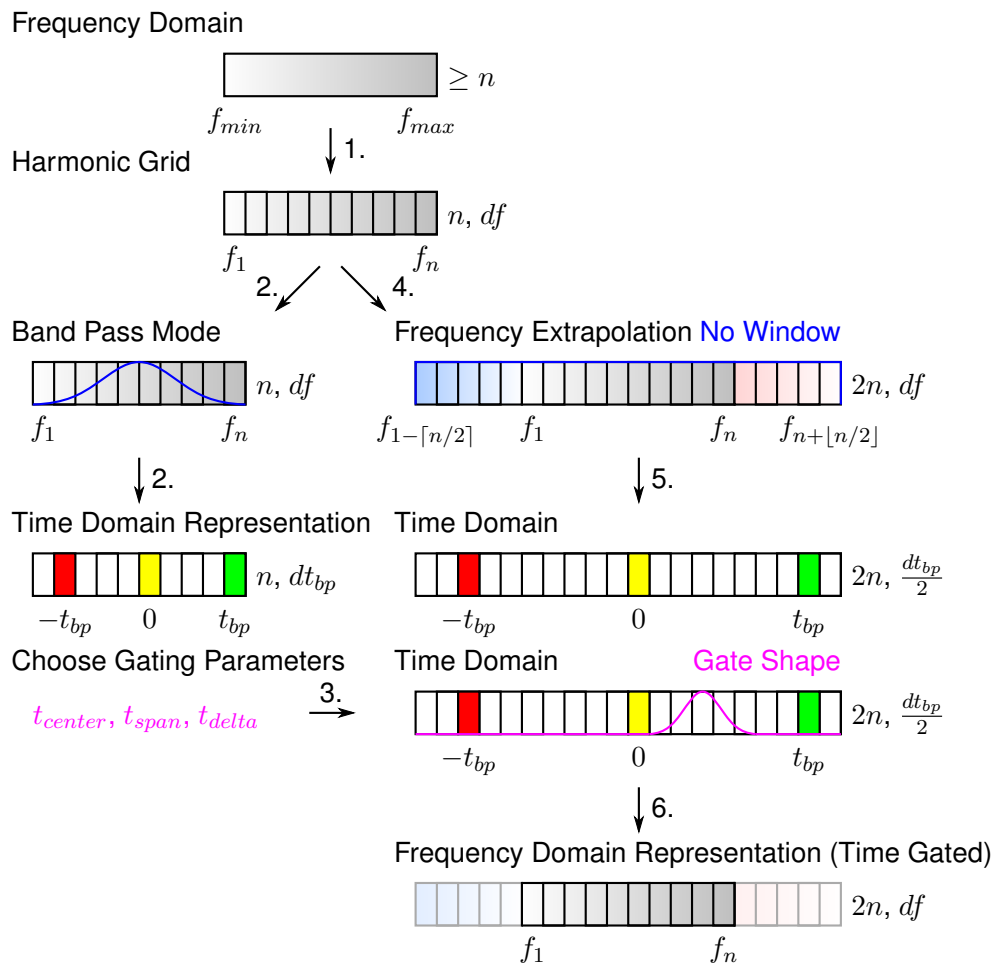


Figure 29: Illustration of time gating using the band pass mode. The steps 2 and 3 on the left side are used for time domain representation and to choose the gating parameters of the gate shape. The steps 4, 5 and 6 on the right side are used for the time gating process.

1. The original data in frequency domain is interpolated on a harmonic grid, see as well section D.1.
2. The data on the harmonic grid is multiplied with the window function and then transformed to time domain using the inverse DFT (discrete fourier transform).



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3. The gating parameters  $t_{center}$ ,  $t_{span}$  and  $t_{delta}$  are chosen. See section D.2.3 for computing the gate shape.
4. The data on the harmonic grid is extrapolated at the lower and higher frequency ranges. The extrapolated points are a mirrored version of the harmonic grid and have large standard uncertainties of  $\pm(1 + |v_{re}|)/\sqrt{3}$  in real part and  $\pm(1 + |v_{im}|)/\sqrt{3}$  in imaginary part where  $|v_p|$  is the absolute value of the part  $p$ . They are used to regularize the value and compute the uncertainty at the boundaries of the time gated data in frequency domain.
5. The extrapolated data in frequency domain is transformed to time domain without applying any window function.
6. The complex data in time domain from step 5 is multiplied with the gate shape and then transformed back to frequency domain. A subset of the frequencies is the result (time gated data in frequency domain).



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### D.2.2 Low Pass Mode

For time gating, using the low pass mode, see figure 30.

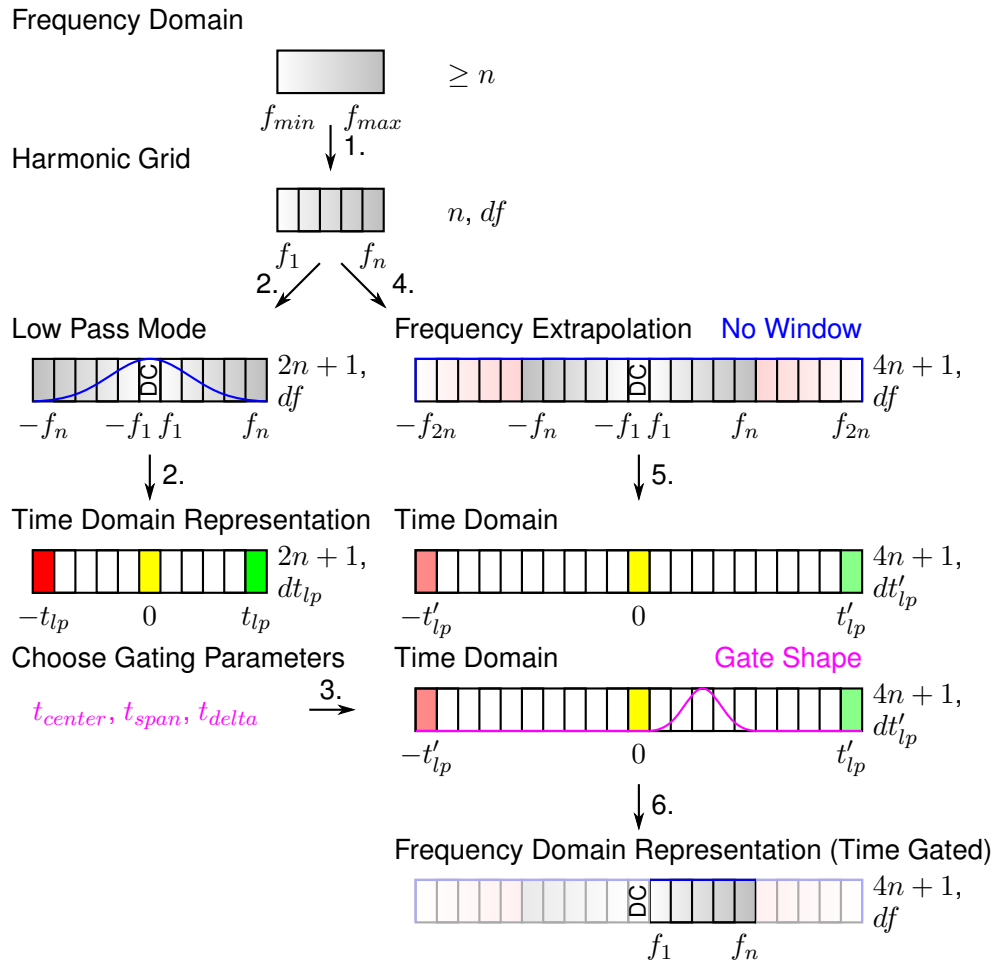


Figure 30: Illustration of time gating using the low pass mode. The steps 2 and 3 on the left side are used for time domain representation and to choose the gating parameters of the gate shape. The steps 4, 5 and 6 on the right side are used for the time gating process.

1. The original data in frequency domain is interpolated on a harmonic grid, see as well section D.1.
2. In the low pass mode, the data of the harmonic grid is mirrored (conjugate complex) to the negative frequencies and the DC point is added if required. The data on the harmonic grid is multiplied with the window function and then transformed to time domain using the inverse DFT (discrete fourier transform).
3. The gating parameters  $t_{center}$ ,  $t_{span}$  and  $t_{delta}$  are chosen. See section D.2.3 for computing the gate shape.



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4. The data on the harmonic grid is extrapolated at the higher negative and positive frequency ranges. The extrapolated points are a mirrored version of the harmonic grid and have large standard uncertainties of  $\pm(1 + |v_{re}|)/\sqrt{3}$  in real part and  $\pm(1 + |v_{im}|)/\sqrt{3}$  in imaginary part where  $|v_p|$  is the absolute value of the part  $p$ . They are used to regularize the value and compute the uncertainty at the boundaries of the time gated data in frequency domain.
5. The extrapolated data in frequency domain is transformed to time domain without applying any window function.  
Note that the time resolution is  $dt'_{lp} = \frac{2n+1}{4n+1} dt_{lp} \approx 0.5 dt_{lp}$ .
6. The complex data in time domain from step 5 is multiplied with the gate shape and then transformed back to frequency domain. A subset of the frequencies is the result (time gated data in frequency domain).



### D.2.3 Gate Shape

The gate shape with the parameters center time  $t_{center}$ , time span  $t_{span}$  and delta time  $t_{delta}$  is shown in figure 31.

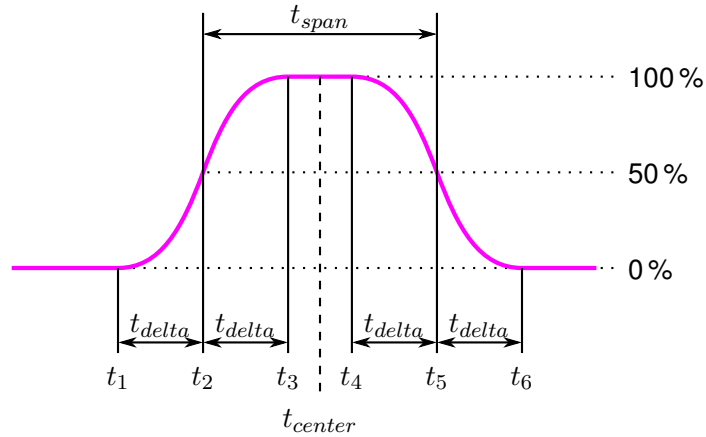


Figure 31: Gate shape

The gate shape  $GS_{bandpass}$  is computed with the following function

$$GS_{bandpass} = \begin{cases} 0 & , t \leq t_1 \\ \text{HannFilter}(t, t_1, t_3 + 2t_{delta}) & , t_1 < t < t_3 \\ 1 & , t_3 \leq t \leq t_4 \\ \text{HannFilter}(t, t_4 - 2t_{delta}, t_6) & , t_4 < t < t_6 \\ 0 & , t_6 \leq t \end{cases} \quad (414)$$

and for a notch gate type

$$GS_{notch} = 1 - GS_{bandpass} \quad (415)$$

where the Hann filter is defined as

$$\text{HannFilter}(t, t_{start}, t_{stop}) = 1 + \frac{\cos\left(2\pi \frac{t - (t_{start} + t_{stop})/2}{t_{stop} - t_{start}}\right)}{2} \quad (416)$$

between  $t_{start} < t < t_{stop}$ . Otherwise it returns 0.



## E METAS UncLib

METAS UncLib [8], [9] is a generic measurement uncertainty calculator that supports the multivariate propagation of measurement uncertainty [7], taking correlations between quantities fully into account, see figure 32.

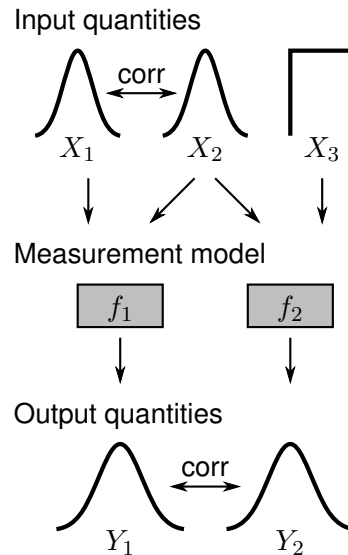


Figure 32: Illustration of multivariate uncertainty propagation: The uncertainties and correlations of basic input quantities  $X_1$ ,  $X_2$ ,  $X_3$  are propagated through a measurement model resulting in uncertainties and correlations of multiple output quantities  $Y_1$ ,  $Y_2$ , see [7].

The user only needs to specify the input quantities  $\mathbf{X}$  with uncertainties, or, more general with an input covariance matrix  $\mathbf{V}_X$ , and the measurement model  $\mathbf{f}$ . The actual propagation of uncertainty is done in the background in an automated way. METAS UncLib keeps automatically track of the derivatives with respect to the input quantities. Formally this means that METAS UncLib computes the output quantities  $\mathbf{Y} = \mathbf{f}(\mathbf{X})$  and the Jacobi matrix  $\mathbf{J}_{YX}$  of  $\mathbf{f}$  that contains the derivatives of the components of  $\mathbf{Y}$  with respect to the components of  $\mathbf{X}$ . On demand METAS UncLib can compute the output covariance matrix  $\mathbf{V}_Y = \mathbf{J}_{YX} \mathbf{V}_X \mathbf{J}_{YX}'$ . Further information about the technique behind METAS UncLib can be found in [9].



## F Small Sample Statistics

The standard uncertainty associated with the final result is multiplied with the coverage factor to obtain an expanded uncertainty with a desired coverage probability (usually 95 %). The calculation of the coverage factor is not straightforward anymore, if the uncertainty contribution due to small sample statistics is significant. The GUM documents [6, 7] are not very consistent in how to treat such a situation, in particular for the multivariate case. We find neither the solution in GUM Supplement 2 with the multivariate t-distribution [7] nor the multivariate generalization of the Welch-Satterthwaite approach [35, 36] satisfactory. The following solution is a self-developed, pragmatic and safe approach to the problem. It will generally overestimate the uncertainty contribution due to small sample statistics.

A series of  $n$  measurements  $(x_1, x_2, x_3, \dots, x_n)$  of a vector quantity with length  $N$  leads to a sample covariance matrix of of the mean

$$\mathbf{S} = \frac{1}{n(n-1)} [(x_1 - \bar{x})(x_1 - \bar{x})' + \dots + (x_n - \bar{x})(x_n - \bar{x})'] . \quad (417)$$

with  $\bar{x}$  being the sample mean vector.

It is assumed that the  $n$  drawings are from a  $N$ -dimensional normal distribution. To obtain a  $p$ -100 % confidence region the covariance matrix needs to be expanded with the following factor squared

$$k_{n,N,p} = \begin{cases} \text{NormalDistCDF}^{-1}\left(\frac{p+1}{2}\right) & , N = 1 \wedge n = \infty \\ \text{StudentTDistCDF}^{-1}\left(n-1, \frac{p+1}{2}\right) & , N = 1 \wedge n < \infty \\ \sqrt{\text{Chi}^2\text{DistCDF}^{-1}(N, p)} & , N > 1 \wedge n = \infty \\ \sqrt{\frac{(n-1)N}{n-N} \text{FDistCDF}^{-1}(n-N, N, p)} & , N > 1 \wedge n < \infty \end{cases} \quad (418)$$

Direct application of this factor can be done, if the covariance matrix is associated with the final result. If it is just an uncertainty contribution among others, which needs to be propagated to the end result, the following practical solution is applied. The sample covariance matrix of the mean  $\mathbf{S}$  is extended using the factor

$$(f_{n,N,p})^2 = \left(\frac{k_{n,N,p}}{k_{\infty,N,p}}\right)^2 , \quad (419)$$

leading to

$$\mathbf{T} = (f_{n,N,p})^2 \mathbf{S} . \quad (420)$$

$\mathbf{T}$  is then used for subsequent uncertainty propagation to the end result. Finally, the covariance matrix associated with the end result is multiplied with  $(k_{\infty,N,p})^2$ .



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Table 11 shows selected  $k$ - and  $f$ -factors for different numbers of measurement repetitions  $n$  and dimensions  $N$ .

Table 11: Coverage Factors

$n$	$k_{n,1,0.95}$	$k_{n,2,0.95}$	$k_{n,8,0.95}$	$f_{n,1,0.95}$	$f_{n,2,0.95}$	$f_{n,8,0.95}$
1						
2	12.7062			6.4829		
3	4.3027	28.2489		2.1953	11.5408	
4	3.1824	7.5498		1.6237	3.0844	
5	2.7764	5.0470		1.4166	2.0619	
6	2.5706	4.1666		1.3115	1.7022	
7	2.4469	3.7265		1.2484	1.5224	
8	2.3646	3.4642		1.2065	1.4153	
9	2.3060	3.2906	123.6466	1.1766	1.3444	31.3989
10	2.2622	3.1674	26.4075	1.1542	1.2940	6.7059
11	2.2281	3.0755	15.3582	1.1368	1.2565	3.9001
12	2.2010	3.0044	11.5284	1.1230	1.2274	2.9275
13	2.1788	2.9477	9.6183	1.1117	1.2042	2.4425
14	2.1604	2.9014	8.4781	1.1022	1.1853	2.1529
15	2.1448	2.8630	7.7209	1.0943	1.1696	1.9606
16	2.1314	2.8305	7.1813	1.0875	1.1564	1.8236
17	2.1199	2.8028	6.7773	1.0816	1.1450	1.7210
18	2.1098	2.7788	6.4633	1.0765	1.1352	1.6413
19	2.1009	2.7578	6.2122	1.0719	1.1267	1.5775
20	2.0930	2.7394	6.0068	1.0679	1.1191	1.5254
50	2.0096	2.5523	4.4984	1.0253	1.0427	1.1423
100	1.9842	2.4983	4.1914	1.0124	1.0206	1.0644
$\infty$	1.9600	2.4477	3.9379	1.0000	1.0000	1.0000

The table helps to determine the improvements in accuracy that can be achieved by increasing the number of measurements. E.g. for  $N = 2$  increasing the measurement repetitions from 4 to 5 leads to a reduction of the uncertainty contribution by approximately 2/3. The selected dimensions are based on the most often used cases:

$N = 1$  : Scalar quantity

$N = 2$  : Complex reflection factor of a 1-port DUT

$N = 8$  : Complex S-matrix of a 2-port DUT



## G Normalized Error

The normalized error is described in [37].

### G.1 Scalar Case

A suitable quantitative verification criteria for a scalar parameter is based on the normalized difference  $\epsilon$ , also referred to as normalized error, between measurement and reference data. The pass condition is  $\epsilon \leq 1$ . The normalized error is expressed as

$$\epsilon = \frac{|d|}{k u(d)}. \quad (421)$$

The quantity  $d$  is the difference between measurement and reference of a scalar quantity.  $u(d)$  is the standard uncertainty of  $d$ .

$k$  is the coverage factor used to expand the uncertainty in denominator in above equation. A value that is often used is  $k = 1.96$ . It expands the uncertainty associated with  $d$  to a 95% coverage interval.  $k = 1$  corresponds to a 68% coverage interval. This verification procedure assumes that the underlying PDF associated with  $d$  is Gaussian.

Remark: tiny differences  $d$  smaller than  $10^{-15}$  are set to zero which yields to a normalized error of 0.

### G.2 Multivariate Case

A straightforward generalization of the scalar criterion is expressed by the matrix equation

$$\epsilon = \frac{1}{k} \sqrt{\mathbf{d} (\mathbf{u}(\mathbf{d}))^{-1} \mathbf{d}'} \quad (422)$$

with prime denoting the transposed. The pass criteria is as well  $\epsilon \leq 1$ .

$\mathbf{d}$  is a row vector, which contains the components of the difference between measurement and reference data.  $\mathbf{u}(\mathbf{d})$  is the covariance matrix associated with  $\mathbf{d}$ .

The coverage factor  $k$  plays the same role as in the scalar case, but the coverage factor  $k$  in the multivariate case is larger than in the scalar case. E.g.: for 95% coverage probability in the two-dimensional case the value  $k = 2,45$  should be used and for  $k = 1$  the coverage probability would be 39%. Again, this assumes that the probability density function associated with the measurement uncertainty of  $\mathbf{d}$  is a multivariate Gaussian one.

The inverse of the covariance matrix is computed using the eigenvalue decomposition

$$(\mathbf{u}(\mathbf{d}))^{-1} = \mathbf{V} \mathbf{D}^{-1} \mathbf{V}' \quad (423)$$

where  $\mathbf{V}$  are the eigenvectors and  $\mathbf{D}^{-1}$  is a diagonal matrix. The elements of the diagonal matrix  $\mathbf{D}^{-1}$  are the inverse of the eigenvalues. The elements of the diagonal matrix  $\mathbf{D}$ , which are smaller in value than  $10^{-15}$  times the largest eigenvalue, are not inverted but set to zero. This is done to avoid numerical problems which may arise from rank deficient covariance matrix  $\mathbf{u}(\mathbf{d})$ .

Remark: tiny differences in the row vector  $\mathbf{d}$  smaller than  $10^{-15}$  are set to zero.



## H Circle Fitting

For circle fitting see figure 33. The algorithm is described in [38].

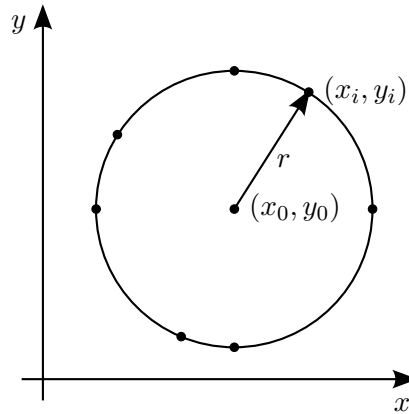


Figure 33: Circle Fitting

The following equation describes a circle

$$r = \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} \quad (424)$$

where  $r$  is the radius,  $x_0, y_0$  are the coordinates of the center of the circle and  $x_i, y_i$  are the coordinates of a point  $i$  on the circle.

One can square the above equation

$$r^2 = (x_i - x_0)^2 + (y_i - y_0)^2 \quad (425)$$

and multiply out the brackets

$$r^2 = x_i^2 - 2x_ix_0 + x_0^2 + y_i^2 - 2y_iy_0 + y_0^2. \quad (426)$$

The above equation can be rearranged as

$$2x_ix_0 + 2y_iy_0 + \underbrace{r^2 - x_0^2 - y_0^2}_{r_0} = x_i^2 + y_i^2. \quad (427)$$

Finally one can setup an over-determined equation system for the unknown coordinates of the center of the circle  $x_0, y_0$  and the unknown intermediate quantity  $r_0$

$$\begin{bmatrix} 2x_1 & 2y_1 & 1 \\ 2x_2 & 2y_2 & 1 \\ \vdots & \vdots & \vdots \\ 2x_n & 2y_n & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ r_0 \end{bmatrix} = \begin{bmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ \vdots \\ x_n^2 + y_n^2 \end{bmatrix} \quad (428)$$

where  $n$  is the number of points on the circle. The radius  $r$  is described with the following equation

$$r = \sqrt{r_0^2 + x_0^2 + y_0^2}. \quad (429)$$



## I Eigenvalue Problem

The most general problem is an over-determined non-linear eigenvalue problem

$$\mathbf{A}_0 \mathbf{v} + \lambda \mathbf{A}_1 \mathbf{v} + \lambda^2 \mathbf{A}_2 \mathbf{v} + \dots + \lambda^n \mathbf{A}_n \mathbf{v} = 0. \quad (430)$$

The over-determined non-linear eigenvalue problem can be transformed to a balanced non-linear eigenvalue problem with order  $m = 2n$  by squaring the over-determined non-linear problem

$$\underbrace{\mathbf{A}_0^* \mathbf{A}_0}_{\mathbf{B}_0} \mathbf{v} + \lambda \underbrace{(\mathbf{A}_0^* \mathbf{A}_1 + \mathbf{A}_1^* \mathbf{A}_0)}_{\mathbf{B}_1} \mathbf{v} + \lambda^2 \underbrace{(\mathbf{A}_0^* \mathbf{A}_2 + \mathbf{A}_1^* \mathbf{A}_1 + \mathbf{A}_2^* \mathbf{A}_0)}_{\mathbf{B}_2} \mathbf{v} + \dots + \lambda^m \underbrace{\mathbf{A}_n^* \mathbf{A}_n}_{\mathbf{B}_m} \mathbf{v} = 0. \quad (431)$$

The operator  $*$  denotes the conjugate transpose.

This non-linear eigenvalue problem can then be rewritten as a linear eigenvalue problem by substitution

$$\underbrace{\begin{bmatrix} \mathbf{B}_0 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & & \mathbf{0} \\ \vdots & \vdots & & \ddots & \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & -\mathbf{I} \end{bmatrix}}_{\mathbf{C}_0} \underbrace{\begin{bmatrix} \mathbf{v} \\ \lambda \mathbf{v} \\ \lambda^2 \mathbf{v} \\ \vdots \\ \lambda^m \mathbf{v} \end{bmatrix}}_{\mathbf{w}} + \lambda \underbrace{\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \dots & \mathbf{B}_m \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & & \mathbf{0} \end{bmatrix}}_{\mathbf{C}_1} \underbrace{\begin{bmatrix} \mathbf{v} \\ \lambda \mathbf{v} \\ \lambda^2 \mathbf{v} \\ \vdots \\ \lambda^m \mathbf{v} \end{bmatrix}}_{\mathbf{w}} = 0 \quad (432)$$

where  $\mathbf{I}$  is the identity matrix.

This linear eigenvalue problem can finally be transformed to a standard eigenvalue problem

$$\underbrace{\mathbf{C}_0^{-1} \mathbf{C}_1}_{\mathbf{D}} \mathbf{w} = \underbrace{-\frac{1}{\lambda}}_{\lambda'} \mathbf{w}. \quad (433)$$

Note that  $\mathbf{C}_0$  is only invertible if  $\mathbf{B}_0$  is invertible. Otherwise the inverse of  $\mathbf{C}_1$  is needed. This yields to the following standard eigenvalue problem

$$\underbrace{\mathbf{C}_1^{-1} \mathbf{C}_0}_{\mathbf{E}} \mathbf{w} = \underbrace{-\lambda}_{\lambda''} \mathbf{w}. \quad (434)$$

The eigenvalue computation with linear uncertainty propagation is described in [39] and it is fully implemented in METAS UncLib [8, 9].



## **J VNA Calibration Model Details**

The following figures show the details of the VNA calibration models:

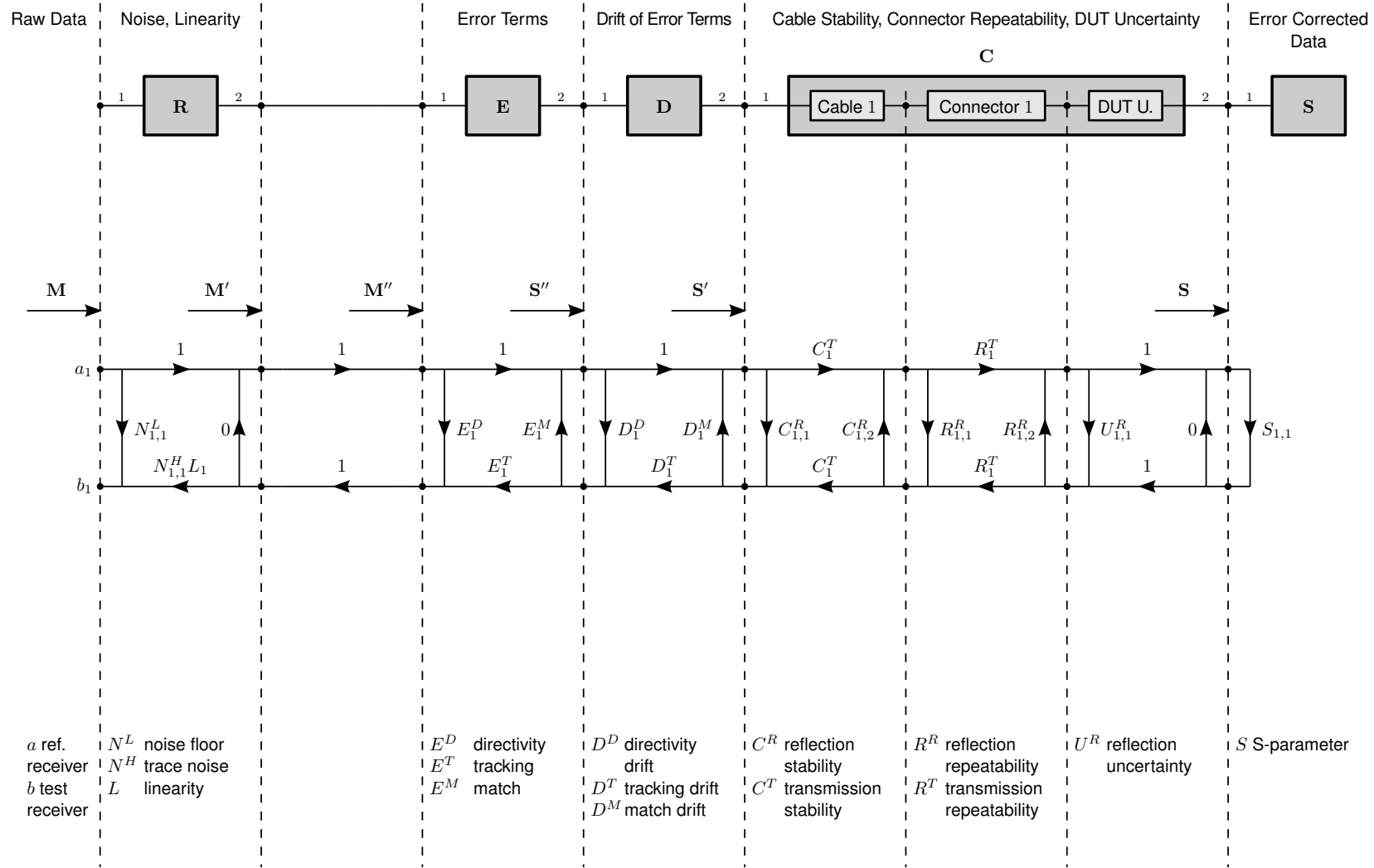
**Figure 34** describes the one-port model.

**Figure 35** describes an one-port calibration.

**Figure 36** describes the difference between the generic and switched two-port model.

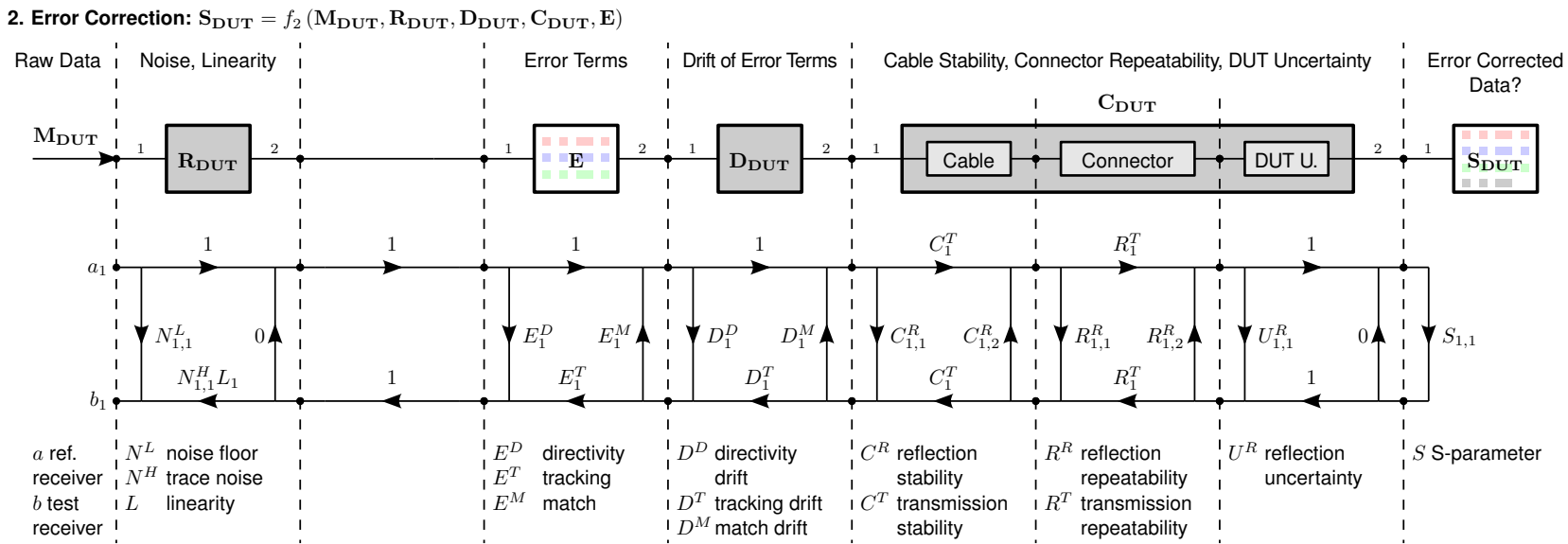
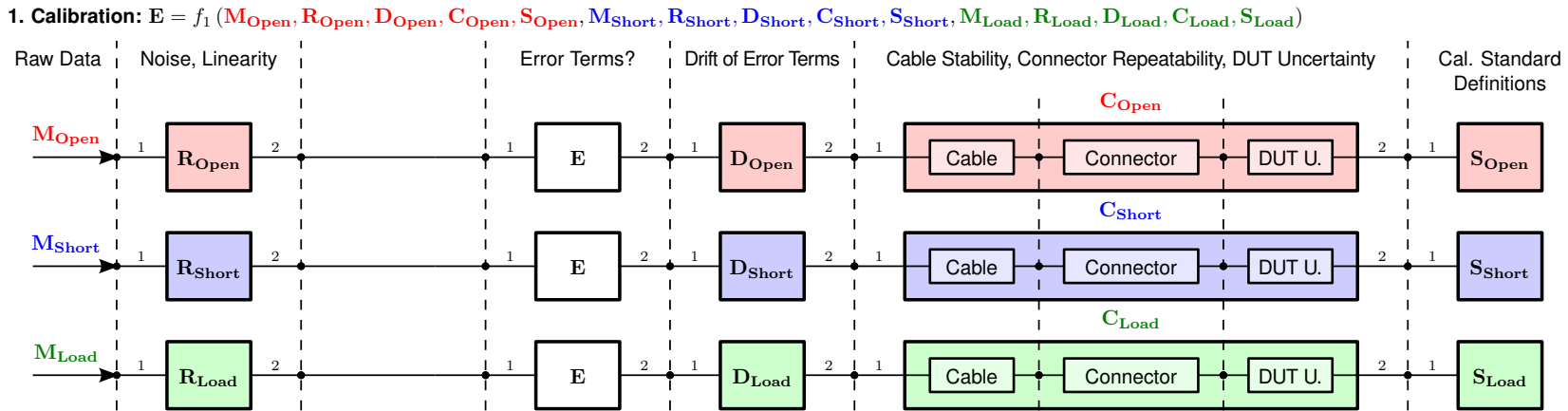
**Figure 37** describes generic multi-port calibration model.

**Figure 38** describes switched multi-port calibration model.



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Figure 34: VNA 1-Port Calibration Model

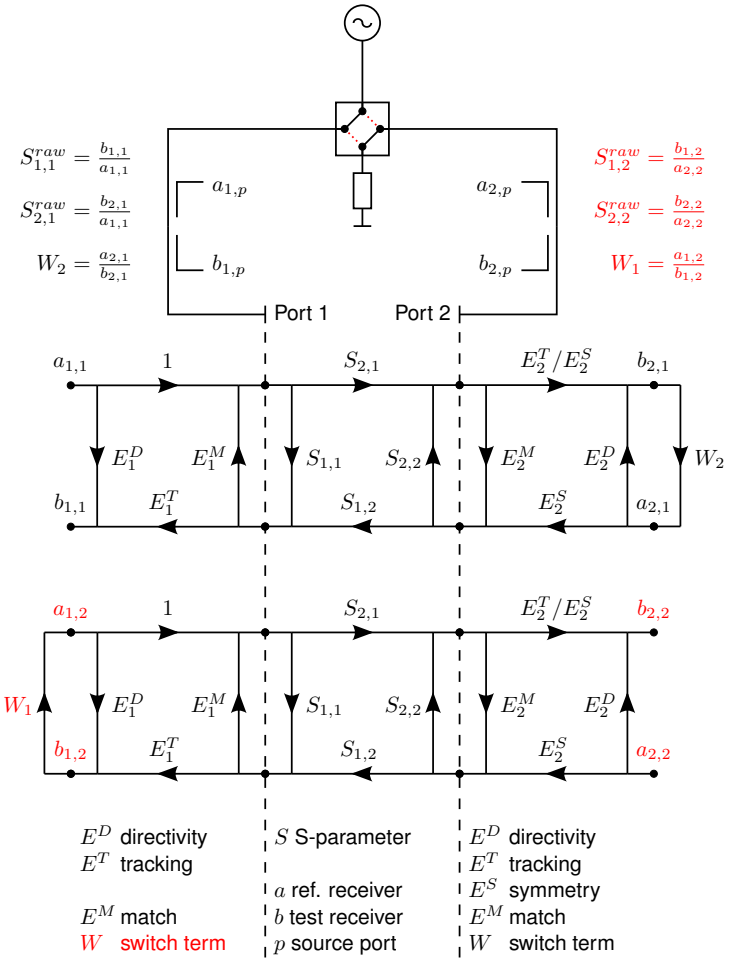


Michael Wollensack METAS - 25.10.2017

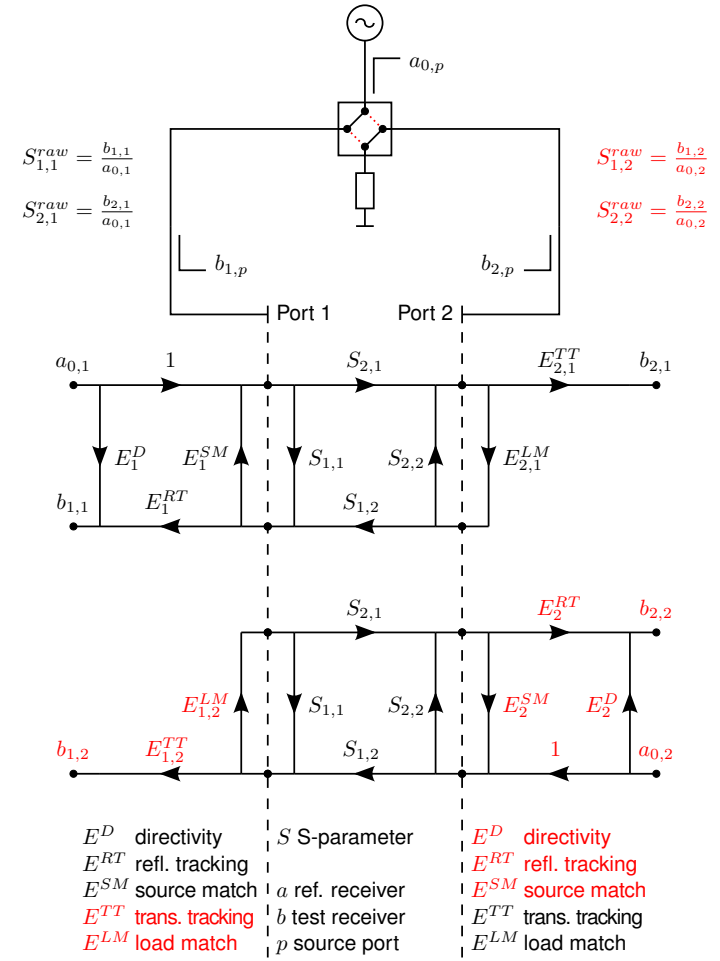
Figure 35: VNA 1-Port Calibration Model cont.



**VNA Generic Calibration Model**  
 QSOLT, Unknown Thru, TRL (4 Receiver, 7 Error and 2 Switch Terms)

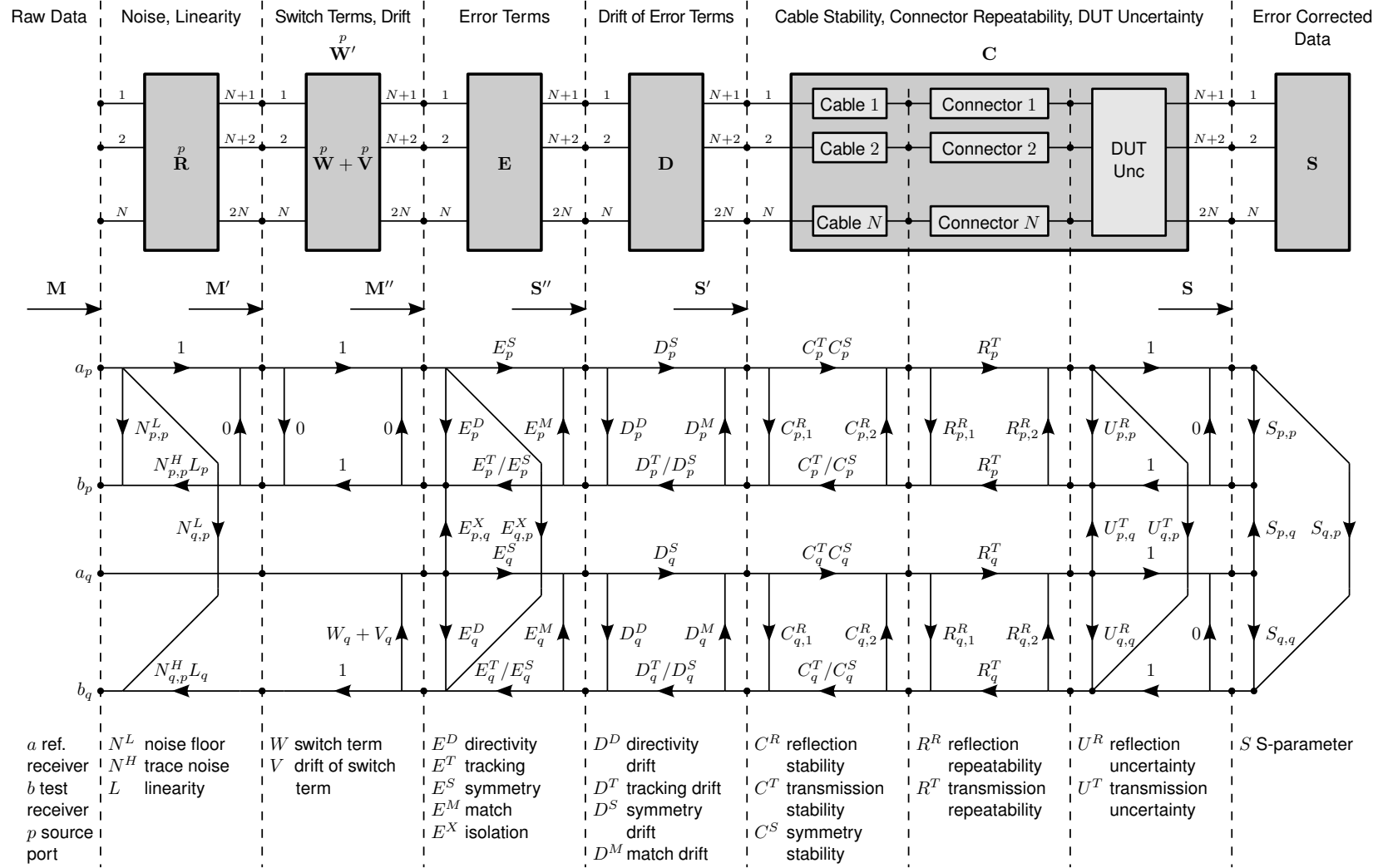


**VNA Switched Calibration Model**  
 SOLT (3 Receiver, 10 Error Terms)



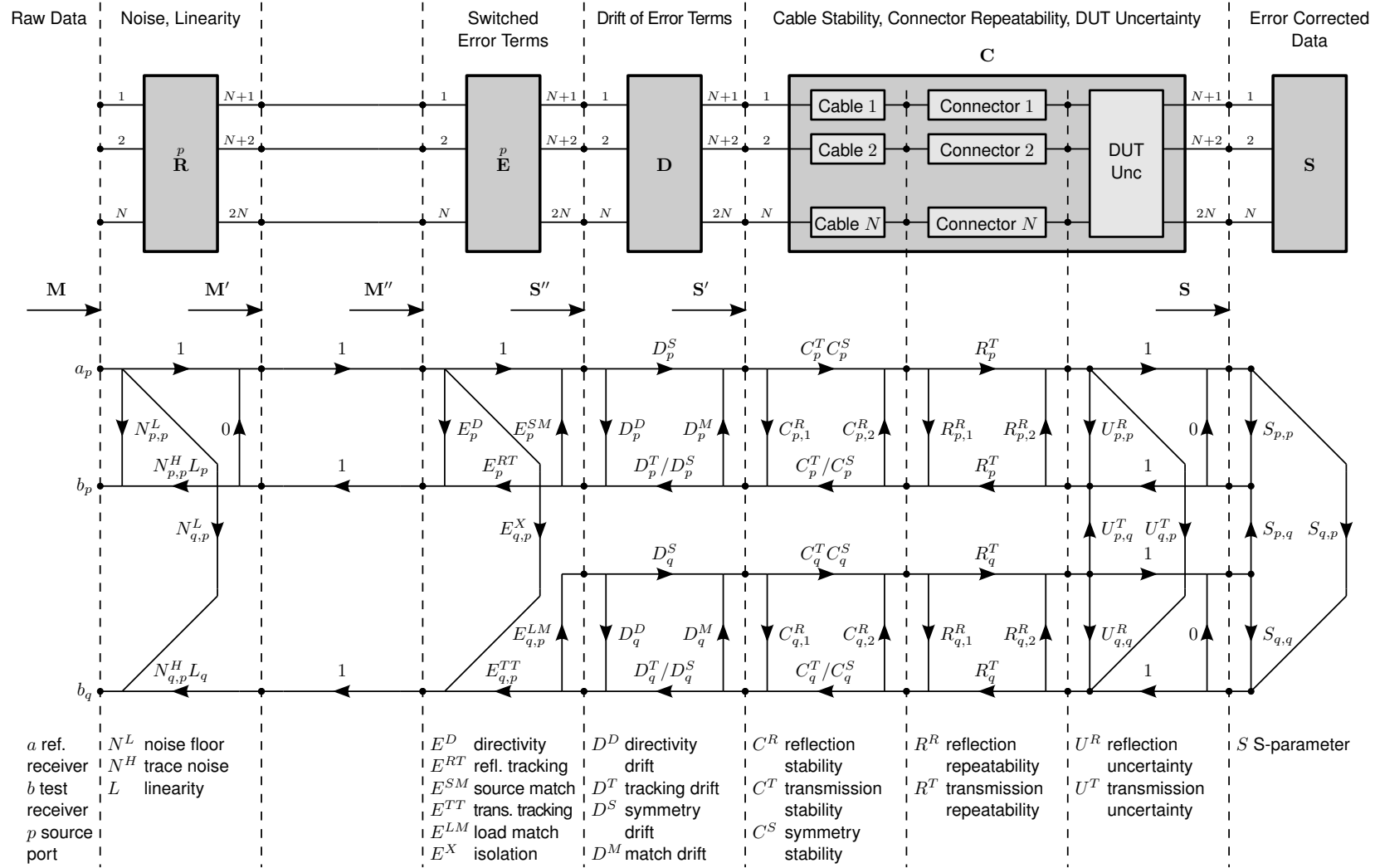
Michael Wollensack METAS - 04.11.2016

Figure 36: VNA 2-Port Calibration Model



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Figure 37: VNA Generic Calibration Model



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Figure 38: VNA Switched Calibration Model



## K Material Parameters

### K.1 Nicholson-Ross-Weir

The algorithm called ‘Nicholson-Ross-Weir’ is described in [40] and [41]. The algorithm is applied to calculate the relative permittivity  $\epsilon_r$  and the relative permeability  $\mu_r$  from transmission  $S_{21}$  and reflection  $S_{11}$  parameters. The results can have poles for low loss samples with thickness of  $> \frac{1}{2}$  wavelength.

The procedure prescribed by the ‘Nicholson-Ross-Weir’ method starts with

$$S_{11} = \frac{\Gamma (1 - T^2)}{1 - \Gamma^2 T^2} \quad (435)$$

and

$$S_{21} = \frac{T (1 - \Gamma^2)}{1 - \Gamma^2 T^2}. \quad (436)$$

The reflection coefficient can be deduced with

$$\Gamma = X \pm \sqrt{X^2 - 1} \quad (437)$$

where  $|\Gamma| < 1$  is the criterion for finding the correct root. In terms of S-parameters  $X$  can be written as

$$X = \frac{S_{11}^2 - S_{21}^2 + 1}{2S_{11}}. \quad (438)$$

Knowing  $\Gamma$  the transmission coefficient is

$$T = \frac{S_{11} + S_{21} - \Gamma}{1 - (S_{11} + S_{21}) \Gamma}. \quad (439)$$

The wavelength  $\Lambda$  in the sample can be deduced with

$$\frac{1}{\Lambda^2} = \left( \frac{\epsilon_r \mu_r}{\lambda_0^2} - \frac{1}{\lambda_c^2} \right) = - \left( \frac{1}{2\pi L} \left( \ln \left( \frac{1}{T} \right) + j2\pi n \right) \right)^2. \quad (440)$$

Here  $\lambda_0$  is the free space wavelength,  $\lambda_c$  is the cut off wavelength and  $L$  is the sample length or thickness. The integer  $n$  is chosen such that the difference between the ‘New Non-Iterative’ (NNI) method and the ‘Direct’ method gets minimized

$$\min_{n \in \mathbb{Z}} \left( (\Re(d))^2 + (\Im(d))^2 \right) \quad (441)$$

with

$$d = \underbrace{\epsilon_{\text{eff}} \mu_{\text{eff}}}_{\text{NNI}} - \underbrace{\frac{\epsilon_{\text{eff}}}{\mu_{\text{eff}}}}_{\text{Direct}} \mu_{\text{guess}}^2 = \frac{\lambda_{og}^2}{\Lambda^2} - \frac{(S_{11} - 1)^2 - S_{21}^2}{(S_{11} + 1)^2 - S_{21}^2} \mu_{\text{guess}}^2 \quad (442)$$

where  $\mu_{\text{guess}}$  is the guess for the relative permeability. The wavelength in an empty cell  $\lambda_{og}$  is

$$\lambda_{og}^2 = \frac{1}{\frac{1}{\lambda_0^2} - \frac{1}{\lambda_c^2}} = \frac{c_0^2}{f_0^2 - f_c^2}. \quad (443)$$



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Here  $f_0$  is the frequency,  $f_c$  is the waveguide cut off frequency and  $c_0$  is the speed of light in vacuum. The effective material parameters are defined as

$$\mu_{\text{eff}} = \frac{\lambda_{og}}{\Lambda} \left( \frac{1 + \Gamma}{1 - \Gamma} \right) \quad (444)$$

and

$$\varepsilon_{\text{eff}} = \frac{\lambda_{og}}{\Lambda} \left( \frac{1 - \Gamma}{1 + \Gamma} \right). \quad (445)$$

The relative permeability is defined as

$$\mu_r = \mu_{\text{eff}}. \quad (446)$$

The relative permittivity is defined as

$$\varepsilon_r = \left( 1 - \frac{f_c^2}{f_0^2} \right) \varepsilon_{\text{eff}} + \frac{f_c^2}{f_0^2} \frac{1}{\mu_{\text{eff}}}. \quad (447)$$

### K.2 NIST Iterative

The algorithm called 'NIST Iterative' is described in [42]. The algorithm is used to calculate the relative permittivity  $\varepsilon_r$  from transmission  $S_{21}$ ,  $S_{12}$  and reflection  $S_{11}$ ,  $S_{22}$  parameters. The relative permeability is assumed as  $\mu_r = 1$ .

The procedure prescribed by the 'NIST Iterative' method starts with the following equations

$$S_{11} = R_1^2 \frac{\Gamma (1 - T^2)}{1 - \Gamma^2 T^2} \quad (448)$$

$$S_{21} = S_{12} = R_1 R_2 \frac{T (1 - \Gamma^2)}{1 - \Gamma^2 T^2} \quad (449)$$

$$S_{22} = R_2^2 \frac{\Gamma (1 - T^2)}{1 - \Gamma^2 T^2} \quad (450)$$

$$R_1 = e^{-\gamma_0 L_1} \quad (451)$$

$$R_2 = e^{-\gamma_0 L_2}. \quad (452)$$

Here  $L_1$  and  $L_2$  are the distances from the calibration reference planes to the sample ends, and  $R_1$  and  $R_2$  are the expressions for reference plane transformation.

The propagation constant in a with material filled sample holder can be written as

$$\gamma = j \sqrt{\frac{(2\pi f_0)^2 \mu_r \varepsilon_r - (2\pi f_c)^2}{c_0^2}} \quad (453)$$

where  $f_0$  is the frequency,  $f_c$  is the waveguide cut off frequency and  $c_0$  is the speed of light in vacuum. The propagation constant in an empty sample holder can be written as

$$\gamma_0 = j \sqrt{\frac{(2\pi f_0)^2 \varepsilon_{\text{air}} - (2\pi f_c)^2}{c_0^2}}. \quad (454)$$



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The reflection coefficient is defined as

$$\Gamma = \frac{\frac{\gamma_0}{\mu_0} - \frac{\gamma}{\mu_0\mu_r}}{\frac{\gamma_0}{\mu_0} + \frac{\gamma}{\mu_0\mu_r}} \quad (455)$$

and for DC

$$\Gamma = \frac{\sqrt{\varepsilon_{\text{air}} \frac{\mu_r}{\varepsilon_r}} - 1}{\sqrt{\varepsilon_{\text{air}} \frac{\mu_r}{\varepsilon_r}} + 1}. \quad (456)$$

The transmission coefficient can be written as

$$T = e^{-\gamma L} \quad (457)$$

where  $L$  is the sample length or thickness. The transmission S-parameter of an empty sample holder is

$$S_{21}^0 = R_1 R_2 e^{-\gamma_0 L}. \quad (458)$$

The optimizer minimizes the following two objective functions

$$f_1 = S_{21} S_{12} - S_{11} S_{22} - e^{-2\gamma_0(L_1+L_2)} \left( \frac{T^2 - \Gamma^2}{1 - \Gamma^2 T^2} \right) \quad (459)$$

and

$$f_2 = \frac{S_{21}}{S_{21}^0} - e^{\gamma_0 L} \left( \frac{T(1 - \Gamma^2)}{1 - \Gamma^2 T^2} \right) \quad (460)$$

by adjusting  $\varepsilon_r$ .

### K.3 New Non-Iterative

The algorithm called 'New Non-Iterative' is based on the 'Nicholson-Ross-Weir' algorithm, see appendix K.1. The algorithm is used to calculate the relative permittivity  $\varepsilon_r$  from transmission  $S_{21}$  and reflection  $S_{11}$  parameters. The relative permeability is assumed as  $\mu_r = \mu_{\text{eff}} = 1$ .

Multiplying equations 444 and 445 from the 'New Non-Iterative' algorithm yields the following effective material parameter

$$\varepsilon_{\text{eff}} \mu_{\text{eff}} = \frac{\lambda_{\text{og}}^2}{\Lambda^2}. \quad (461)$$

Note that this solution has no poles. The relative permittivity is defined as

$$\varepsilon_r = \left( 1 - \frac{f_c^2}{f_0^2} \right) \frac{\varepsilon_{\text{eff}} \mu_{\text{eff}}}{\mu_{\text{eff}}} + \frac{f_c^2}{f_0^2} \frac{1}{\mu_{\text{eff}}}. \quad (462)$$

Here  $f_0$  is the frequency and  $f_c$  is the waveguide cut off frequency.



#### K.4 Direct

The algorithm called 'Direct' is based on the 'Nicholson-Ross-Weir' algorithm, see appendix K.1. The algorithm is used to calculate the relative permittivity  $\varepsilon_r$

$$\varepsilon_r = \left(1 - \frac{f_c^2}{f_0^2}\right) \frac{\varepsilon_{\text{eff}}}{\mu_{\text{eff}}} \mu_{\text{eff}} + \frac{f_c^2}{f_0^2} \frac{1}{\mu_{\text{eff}}} \quad (463)$$

with

$$\frac{\varepsilon_{\text{eff}}}{\mu_{\text{eff}}} = \frac{(S_{11} - 1)^2 - S_{21}^2}{(S_{11} + 1)^2 - S_{21}^2} \quad (464)$$

from transmission  $S_{21}$  and reflection  $S_{11}$  parameters.  $f_0$  is the frequency and  $f_c$  is the waveguide cut off frequency. The relative permeability is assumed as  $\mu_r = \mu_{\text{eff}} = 1$ . The result can have poles for low loss samples with thickness of  $> \frac{1}{2}$  wavelength.

#### K.5 Transmission Only

The algorithm called 'Transmission Only' is described in [43]. This method utilizes the  $S_{21}$  parameter to derive the transmission coefficient  $T$  and to extract the permittivity  $\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$ . Here  $T$  is a function of the sample length or thickness  $L$  and its permittivity and permeability. In this method the relative permeability is assumed as  $\mu_r = 1$ . For the materials which are not very lossy  $\varepsilon_r''/\varepsilon_r' = \tan \delta < 0.1$ ,  $T$  can be separated directly to phase and magnitude parts

$$T \approx e^{\left(-j\frac{2\pi f_0 L}{v} \sqrt{\varepsilon_{\text{eff}}'}\right)} e^{\left(-\frac{2\pi f_0 L}{2v\sqrt{\varepsilon_{\text{eff}}'}} \varepsilon_{\text{eff}}''\right)} \quad (465)$$

with the phase velocity

$$v = \frac{c_0}{\sqrt{1 - \frac{f_c^2}{f_0^2}}} \quad (466)$$

where  $\varepsilon_{\text{eff}}$  is the effective permittivity,  $f_0$  is the frequency,  $f_c$  is the waveguide cut off frequency and  $c_0$  is the speed of light in vacuum. From equation 465 one can compute the real part of the effective permittivity

$$\varepsilon_{\text{eff}}' = \left(-\frac{v}{2\pi f_0 L} \arg(T)\right)^2 \quad (467)$$

where  $\arg(T)$  is the unwrapped phase of  $T$ . The imaginary part of the effective permittivity is

$$\varepsilon_{\text{eff}}'' = -\frac{2v\sqrt{\varepsilon_{\text{eff}}'}}{2\pi f_0 L} \ln(|T|). \quad (468)$$

From the effective permittivity  $\varepsilon_{\text{eff}} = \varepsilon_{\text{eff}}' - j\varepsilon_{\text{eff}}''$  one can compute the relative permittivity using the following equation

$$\varepsilon_r = \left(1 - \frac{f_c^2}{f_0^2}\right) \varepsilon_{\text{eff}} + \frac{f_c^2}{f_0^2} \frac{1}{\mu_{\text{eff}}} \quad (469)$$

where the relative permeability is assumed as  $\mu_r = \mu_{\text{eff}} = 1$ .



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Using an unknown reflection coefficient  $\Gamma$  the following steps are needed to compute the relative permittivity:

1. One computes a guess of the real part of the relative permittivity  $\varepsilon_r'$  using the above equations 467 and 469 at frequency points where the phase of  $S_{21}$  is a multiple of  $\pi$ . At these points the unwrapped phase of  $T$  is equal to the unwrapped phase of  $S_{21}$ . The guess of the imaginary part of the relative permittivity is assumed as  $\varepsilon_r'' = 0$ . This guess for the relative permittivity is assumed to be constant for all frequency points.
2. The guess of the relative permittivity is used to compute the reflection coefficient, which is defined as

$$\Gamma = \frac{\sqrt{f_0^2 - f_c^2} - \sqrt{f_0^2 \varepsilon_r - f_c^2}}{\sqrt{f_0^2 - f_c^2} + \sqrt{f_0^2 \varepsilon_r - f_c^2}} \quad (470)$$

3. One solves the following equation for  $T$

$$S_{21} = \frac{T(1 - \Gamma^2)}{1 - \Gamma^2 T^2} \quad (471)$$

using the reflection coefficient  $\Gamma$  from the step before and the  $S_{21}$  parameter. This yields into a quadratic equation

$$S_{21} \Gamma^2 T^2 + (1 - \Gamma^2) T - S_{21} = 0. \quad (472)$$

The solution of  $T$  which has the smaller magnitude is chosen.

4. One computes the relative permittivity  $\varepsilon_r$  for each frequency point using  $T$  from step three and equations 467, 468 and 469.

The unwrapped phase of  $S_{21}$  has an unknown offset of  $2\pi n$ . The integer  $n$  is chosen such that the standard deviation of the different frequency points of  $\varepsilon_r'$  of the above steps gets minimized.



### K.6 On-Wafer Heinrich

The algorithm called 'On-Wafer Heinrich' is using optimization to minimize the difference between the error correct S-parameters  $\mathbf{S}$  of the measurement and the modelled S-parameters  $\mathbf{H}$  of an on-wafer line standard.

The characteristic impedance  $Z_0$  and the propagation constant  $\gamma$  are computed using the modified Heinrich model which is described in [21] and [22], the Schnieder model which is described in [23], the Gold model which is described in [24], [25] and [26] or the Phung model which is described in [27]. They are a function of  $\epsilon_r$ . For details see section 6.15.

#### K.6.1 Reflection and Transmission

The modelled S-parameters  $\mathbf{H}$  of an on-wafer line standard are a function of  $Z_0$  and  $\gamma l$  where  $l$  is the length. For details see section A.4.

By adjusting  $\epsilon_r$  the optimizer minimizes the following objective functions

$$f_1 = \operatorname{Re}(\mathbf{S}_{11}) - \operatorname{Re}(\mathbf{H}_{11}(\epsilon_r)) \quad (473)$$

$$f_2 = \operatorname{Im}(\mathbf{S}_{11}) - \operatorname{Im}(\mathbf{H}_{11}(\epsilon_r)) \quad (474)$$

$$f_3 = \operatorname{Re}(\mathbf{S}_{21}) - \operatorname{Re}(\mathbf{H}_{21}(\epsilon_r)) \quad (475)$$

$$f_4 = \operatorname{Im}(\mathbf{S}_{21}) - \operatorname{Im}(\mathbf{H}_{21}(\epsilon_r)) \quad (476)$$

$$f_5 = \operatorname{Re}(\mathbf{S}_{12}) - \operatorname{Re}(\mathbf{H}_{12}(\epsilon_r)) \quad (477)$$

$$f_6 = \operatorname{Im}(\mathbf{S}_{12}) - \operatorname{Im}(\mathbf{H}_{12}(\epsilon_r)) \quad (478)$$

$$f_7 = \operatorname{Re}(\mathbf{S}_{22}) - \operatorname{Re}(\mathbf{H}_{22}(\epsilon_r)) \quad (479)$$

$$f_8 = \operatorname{Im}(\mathbf{S}_{22}) - \operatorname{Im}(\mathbf{H}_{22}(\epsilon_r)). \quad (480)$$

#### K.6.2 Transmission Only

The modelled S-parameters  $\mathbf{H}$  with  $Z_r = Z_0$  of an on-wafer line standard are a function of  $\gamma l$  and the length  $l$ .

By adjusting  $\epsilon_r$  the optimizer minimizes the following objective functions

$$f_1 = \operatorname{Re}(\mathbf{S}_{21}) - \operatorname{Re}(\mathbf{H}_{21}(\epsilon_r)) \quad (481)$$

$$f_2 = \operatorname{Im}(\mathbf{S}_{21}) - \operatorname{Im}(\mathbf{H}_{21}(\epsilon_r)) \quad (482)$$

$$f_3 = \operatorname{Re}(\mathbf{S}_{12}) - \operatorname{Re}(\mathbf{H}_{12}(\epsilon_r)) \quad (483)$$

$$f_4 = \operatorname{Im}(\mathbf{S}_{12}) - \operatorname{Im}(\mathbf{H}_{12}(\epsilon_r)) \quad (484)$$

with

$$\mathbf{H}_{11} = 0 \quad (485)$$

$$\mathbf{H}_{21} = e^{-\gamma(\epsilon_r)l} \quad (486)$$

$$\mathbf{H}_{12} = e^{-\gamma(\epsilon_r)l} \quad (487)$$

$$\mathbf{H}_{22} = 0. \quad (488)$$



### L Noise Parameters

Noise parameters are described in [44].

#### L.1 Noise Temperature

The following equation describes the available noise power  $P_a$  in W divided by the equivalent noise bandwidth  $B$  in Hz

$$\frac{P_a}{B} = \frac{hf}{e^{\frac{hf}{T_p k}} - 1} \quad (489)$$

where  $h$  is the Plank constant in Js,  $f$  is the frequency in Hz,  $T_p$  is the physical temperature of a resistor in kelvin and  $k$  is the Boltzmann constant in J/K.

The available noise temperature is defined as

$$T_a = \frac{1}{k} \frac{P_a}{B} = \frac{1}{k} \frac{hf}{e^{\frac{hf}{T_p k}} - 1}. \quad (490)$$

The effective noise temperature is defined as

$$T_{ne} = T_a (1 - |\Gamma_s|^2) \quad (491)$$

where  $\Gamma_s$  is the reflection coefficient of the source.

#### L.2 Noise Factor

The following equation describes the noise factor

$$F = \frac{T_e}{T_0} + 1 \quad (492)$$

where  $T_e$  is the effective input noise temperature in kelvin and  $T_0$  is the reference temperature which is normally set to 290 K.

The inverse function of the above equation is

$$T_e = T_0(F - 1). \quad (493)$$

#### L.3 Noise Figure

The following equation describes the noise figure

$$NF = 10 \log(F) \quad (494)$$

where  $F$  is the noise factor.

The inverse function of the above equation is

$$F = 10^{\frac{NF}{10}}. \quad (495)$$



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### L.4 Excess Noise Ratio

The following equation describes the excess noise ratio in dB

$$ENR = 10 \log \left( \frac{T_h - T_c}{T_0} \right) \quad (496)$$

where  $T_h$  is the hot noise temperature of the noise source and  $T_c$  is the cold noise temperature of the noise source. By definition  $T_c$  is set to  $T_0$ .

### L.5 Friis Formula

Friis's formula is used to calculate the total noise factor of a cascade of stages, each with its own noise factor  $F_i$  and power gain  $G_i$ . The total noise factor is given as

$$F_{total} = F_1 + \frac{F_2 - 1}{G_1} + \frac{F_3 - 1}{G_1 G_2} + \frac{F_4 - 1}{G_1 G_2 G_3} + \dots + \frac{F_n - 1}{G_1 G_2 \dots G_{n-1}} \quad (497)$$

where  $n$  is the number of stages.

Friis's formula can be equivalently expressed in terms of noise temperature

$$T_{eq} = T_1 + \frac{T_2}{G_1} + \frac{T_3}{G_1 G_2} + \frac{T_4}{G_1 G_2 G_3} + \dots + \frac{T_n}{G_1 G_2 \dots G_{n-1}}. \quad (498)$$

#### L.5.1 Gain

The following equation describes the available gain

$$G_a = \frac{(1 - |\Gamma_s|^2) |S_{21}|^2}{|1 - S_{11}\Gamma_s|^2 \left( 1 - \left| S_{22} + \frac{S_{12}S_{21}\Gamma_s}{1 - S_{11}\Gamma_s} \right|^2 \right)} \quad (499)$$

where  $\Gamma_s$  is the source reflection,  $S_{11}$  is the input reflection and  $S_{22}$  is the output reflection,  $S_{21}$  and  $S_{12}$  are the transmission S-parameters.

### L.6 IEEE Noise Parameters

The following equation describes the effective input noise temperature in kelvin

$$T_e = T_{e,min} + \frac{4R_n T_0}{Z_r} \left( \frac{|\Gamma_{opt} - \Gamma_s|^2}{|1 + \Gamma_{opt}|^2 (1 - |\Gamma_s|^2)} \right) \quad (500)$$

where  $T_{e,min}$  in kelvin,  $R_n$  in Ohm and  $\Gamma_{opt}$  are the IEEE noise parameters.  $T_0$  is the reference temperature which is normally set to 290 K.  $Z_r$  is the reference impedance in Ohm and  $\Gamma_s$  is the source reflection.



## L.7 NIST Noise Parameters

NIST noise parameters are described in [45]. The following equation describes the effective input noise temperature in kelvin

$$T_e = \frac{|\Gamma_s|^2}{1 - |\Gamma_s|^2} X_1 + \frac{|1 - \Gamma_s S_{11}|^2}{1 - |\Gamma_s|^2} X_2 + \frac{2}{1 - |\Gamma_s|^2} \operatorname{Re}((1 - \Gamma_s S_{11})^* \Gamma_s X_{12}) \quad (501)$$

where  $X_1$ ,  $X_2$  and  $X_{12}$  are the NIST noise parameters in kelvin.  $S_{11}$  is the input reflection and  $\Gamma_s$  is the source reflection. The operator  $*$  denotes the conjugate transpose.

### L.7.1 Solving NIST Noise Parameters

The NIST noise parameters can be solved when having at least a set of four different effective input noise temperatures  $T_{e,i}$  and source reflections  $\Gamma_{s,i}$  where  $i$  is the measurement index.

In a first step one can rewrite the equation 501 as

$$T_e (1 - |\Gamma_s|^2) = |\Gamma_s|^2 X_1 + |1 - \Gamma_s S_{11}|^2 X_2 + 2 \operatorname{Re}(m X_{12}) \quad (502)$$

with

$$m = (1 - \Gamma_s S_{11})^* \Gamma_s. \quad (503)$$

In a second step one can split the noise parameter  $X_{12} = X_{12}^{Re} + jX_{12}^{Im}$  into its real and imaginary parts and rewrite the equation 502 as

$$T_e (1 - |\Gamma_s|^2) = |\Gamma_s|^2 X_1 + |1 - \Gamma_s S_{11}|^2 X_2 + 2 \operatorname{Re}(m) X_{12}^{Re} - 2 \operatorname{Im}(m) X_{12}^{Im}. \quad (504)$$

Finally one can setup an over-determined system for the unknown NIST noise parameters  $X_1$ ,  $X_2$  and  $X_{12}$

$$\begin{bmatrix} |\Gamma_{s,1}|^2 & |1 - \Gamma_{s,1} S_{11}|^2 & 2 \operatorname{Re}(m_1) & -2 \operatorname{Im}(m_1) \\ |\Gamma_{s,2}|^2 & |1 - \Gamma_{s,2} S_{11}|^2 & 2 \operatorname{Re}(m_2) & -2 \operatorname{Im}(m_2) \\ |\Gamma_{s,3}|^2 & |1 - \Gamma_{s,3} S_{11}|^2 & 2 \operatorname{Re}(m_3) & -2 \operatorname{Im}(m_3) \\ \vdots & \vdots & \vdots & \vdots \\ |\Gamma_{s,n}|^2 & |1 - \Gamma_{s,n} S_{11}|^2 & 2 \operatorname{Re}(m_n) & -2 \operatorname{Im}(m_n) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_{12}^{Re} \\ X_{12}^{Im} \end{bmatrix} = \begin{bmatrix} T_{e,1} (1 - |\Gamma_{s,1}|^2) \\ T_{e,2} (1 - |\Gamma_{s,2}|^2) \\ T_{e,3} (1 - |\Gamma_{s,3}|^2) \\ \vdots \\ T_{e,n} (1 - |\Gamma_{s,n}|^2) \end{bmatrix} \quad (505)$$

with

$$m_i = (1 - \Gamma_{s,i} S_{11})^* \Gamma_{s,i} \quad (506)$$

where  $n$  is the number of different effective input noise temperatures  $T_{e,i}$  and source reflections  $\Gamma_{s,i}$ .

The measured effective noise temperature  $T_{ne,i}^b$  is given by the following equation

$$T_{ne,i}^b = (T_{e,i} + T_{a,i}^s) G_{a,i} \quad (507)$$

where  $T_{e,i}$  is the effective input noise temperature,  $T_{a,i}^s$  is the available noise temperature of the source and  $G_{a,i}$  is the available gain.

One can rewrite the equation 507 as

$$T_{e,i} = \frac{T_{ne,i}^b}{G_{a,i}} - T_{a,i}^s \quad (508)$$



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and multiply it with  $(1 - |\Gamma_{s,i}|^2)$  which yields to

$$T_{e,i} (1 - |\Gamma_{s,i}|^2) = T_{ne,i}^b \frac{(1 - |\Gamma_{s,i}|^2)}{G_{a,i}} - T_{ne,i}^s \quad (509)$$

with

$$T_{ne,i}^s = T_{a,i}^s (1 - |\Gamma_{s,i}|^2) \quad (510)$$

where  $T_{ne,i}^s$  is the effective noise temperature of the source. Equation 509 is used to compute the right side of the equation system 505.

### L.8 Noise Correlation Matrix

Noise waves and the noise correlation matrix are described in [45] and [46].

The S-parameters  $\mathbf{S}$  and the noise waves  $\mathbf{c}$  are related by

$$\mathbf{b} = \mathbf{S}\mathbf{a} + \mathbf{c} \quad (511)$$

where  $\mathbf{b}$  is a column vector for the outgoing waves and  $\mathbf{a}$  is a column vector for the incident waves.

Characterization of the random noise wave amplitudes is made in terms of a noise correlation matrix

$$\mathbf{N} = \overline{\mathbf{c}\mathbf{c}^*} \quad (512)$$

where the operator  $*$  denotes the conjugate transpose and the overbar the correlation product. For each noise wave  $c_i$  the correlation matrix yields values for the noise power deliverable to the terminations, given in terms of the statistical expectation of  $c_i$ , and written as  $\overline{|c_i|^2}$ , as well as a correlation product with each of the other ports, given by  $\overline{c_i c_j^*}$ .

An  $N$ -port linear component's signal and noise properties are completely described by a  $N \times N$  S-parameter matrix  $\mathbf{S}$  and a  $N \times N$  noise correlation matrix  $\mathbf{N}$ .

#### L.8.1 Noise Correlation Matrix from Passive S-Parameters

Noise parameters of passive devices under test are described in [46].

The following equation describes the noise correlation matrix of passive  $N$ -ports

$$\mathbf{N} = kT_a (\mathbf{I} - \mathbf{S}\mathbf{S}^*) \quad (513)$$

where  $k$  is the Boltzmann constant in J/K,  $\mathbf{I}$  is the identity matrix and  $\mathbf{S}$  are the S-parameters of the  $N$ -port. The operator  $*$  denotes the conjugate transpose.

One can compute the available noise temperature  $T_a$  in K using the physical temperature  $T_p$  in K and the frequency  $f$  in Hz, see equation 490.



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### L.9 Noise Parameters Tools

#### L.9.1 IEEE to NIST Noise Parameters

The following equations describe the conversion from IEEE noise parameters  $T_{e,min}$ ,  $R_n$  and  $\Gamma_{opt}$  to NIST noise parameters

$$X_1 = T_{e,min} (|S_{11}|^2 - 1) + \frac{4R_n T_0}{Zr} \frac{|1 - S_{11}\Gamma_{opt}|^2}{|1 + \Gamma_{opt}|^2} \quad (514)$$

$$X_2 = T_{e,min} + \frac{4R_n T_0}{Zr} \frac{|\Gamma_{opt}|^2}{|1 + \Gamma_{opt}|^2} \quad (515)$$

$$X_{12} = S_{11}T_{e,min} - \frac{4R_n T_0}{Zr} \frac{\Gamma_{opt}^*(1 - S_{11}\Gamma_{opt})}{|1 + \Gamma_{opt}|^2} \quad (516)$$

where  $S_{11}$  is the input reflection. The operator \* denotes the conjugate transpose.

#### L.9.2 NIST to IEEE Noise Parameters

The following equations describe the conversion from NIST noise parameters  $X_1$ ,  $X_2$  and  $X_{12}$  to IEEE noise parameters

$$R_n = \frac{Zr}{4T_0} (X_1 + |1 + S_{11}|^2 X_2 - 2 \operatorname{Re}((1 + S_{11})^* X_{12})) \quad (517)$$

$$\Gamma_{opt} = \begin{cases} 0 & , R_n = 0 \\ \text{NaN} & , R_n \neq 0 \wedge |p| < 2 \\ \frac{\rho}{2} \left(1 - \sqrt{1 - \frac{4}{|\rho|^2}}\right) & , R_n \neq 0 \wedge |p| \geq 2 \end{cases} \quad (518)$$

$$T_{e,min} = \frac{X_2 - |\Gamma_{opt}|^2 (X_1 + |S_{11}|^2 X_2 - 2 \operatorname{Re}(S_{11}^* X_{12}))}{1 + |\Gamma_{opt}|^2} \quad (519)$$

with

$$\rho = \frac{X_2 (1 + |S_{11}|^2) + X_1 - 2 \operatorname{Re}(S_{11}^* X_{12})}{X_2 S_{11} - X_{12}} \quad (520)$$

where  $S_{11}$  is the input reflection. The operator \* denotes the conjugate transpose.

#### L.9.3 NIST Noise Parameters to Noise Correlation Matrix

The following equations describe the conversion from NIST noise parameters  $X_1$ ,  $X_2$  and  $X_{12}$  to the noise correlation matrix

$$\mathbf{N} = k \begin{bmatrix} X_1 & X_{12} S_{21}^* \\ X_{12}^* S_{21} & X_2 |S_{21}|^2 \end{bmatrix} \quad (521)$$

where  $k$  is the Boltzmann constant in J/K and  $S_{21}$  is the forward transmission. The operator \* denotes the conjugate transpose.



**L.9.4 Noise Correlation Matrix to NIST Noise Parameter**

The following equations describe the conversion from the noise correlation matrix  $\mathbf{N}$  to NIST noise parameters for a two-port

$$X_1 = \frac{\text{Re}(\mathbf{N}_{1,1})}{k} \tag{522}$$

$$X_2 = \frac{\text{Re}(\mathbf{N}_{2,2})}{k|S_{21}|^2} \tag{523}$$

$$X_{12} = \frac{\mathbf{N}_{1,2}}{kS_{21}^*} \tag{524}$$

where  $k$  is the Boltzmann constant in J/K and  $S_{21}$  is the forward transmission. The operator  $*$  denotes the conjugate transpose.

**L.9.5 Cascading of a 2N-Port and a N-Port**

Cascading of S-parameters of a  $2N$ -port and a  $N$ -port is described in section A.1.1. This section describes an extension for cascading S-parameters and noise correlation matrices.

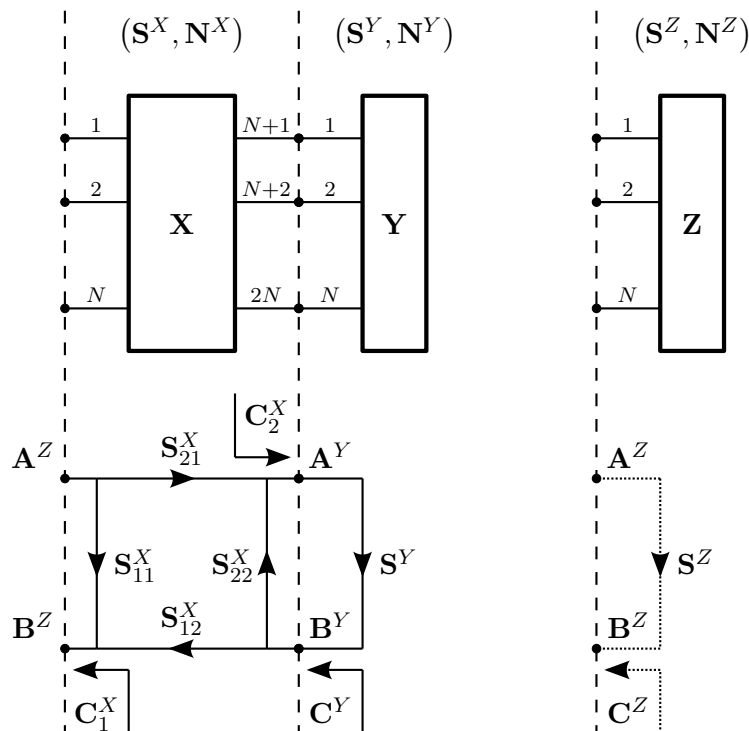


Figure 39: Cascading of a  $2N$ -port ( $\mathbf{X}$ ) and a  $N$ -port ( $\mathbf{Y}$ ) including noise parameters

One can use the following equation to cascade the S-parameters  $\mathbf{S}^X$  and  $\mathbf{S}^Y$  of the  $2N$ -port  $\mathbf{X}$  and the  $N$ -port  $\mathbf{Y}$ . The result will be the S-parameters  $\mathbf{S}^Z$  of the  $N$ -port  $\mathbf{Z}$

$$\mathbf{S}^Z = \mathbf{S}_{11}^X + \mathbf{S}_{12}^X (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} \mathbf{S}^Y \mathbf{S}_{21}^X \tag{525}$$



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with

$$\mathbf{S}^X = \begin{bmatrix} \mathbf{S}_{11}^X & \mathbf{S}_{12}^X \\ \mathbf{S}_{21}^X & \mathbf{S}_{22}^X \end{bmatrix} \quad (526)$$

where  $\mathbf{I}$  is the identity matrix.

One can use the following equation to cascade the noise waves  $\mathbf{C}^X$  and  $\mathbf{C}^Y$  of the  $2N$ -port  $\mathbf{X}$  and the  $N$ -port  $\mathbf{Y}$ . The result will be the noise waves  $\mathbf{C}^Z$  of the  $N$ -port  $\mathbf{Z}$

$$\mathbf{C}^Z = \mathbf{C}_1^X + \mathbf{S}_{12}^X (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} \mathbf{C}^Y + \mathbf{S}_{12}^X (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} \mathbf{S}^Y \mathbf{C}_2^X \quad (527)$$

with

$$\mathbf{C}^X = \begin{bmatrix} \mathbf{C}_1^X \\ \mathbf{C}_2^X \end{bmatrix}. \quad (528)$$

One can introduce the following two intermediate quantities

$$\mathbf{T}_1 = \mathbf{S}_{12}^X (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} \quad (529)$$

$$\mathbf{T}_2 = \mathbf{S}_{12}^X (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} \mathbf{S}^Y = \mathbf{T}_1 \mathbf{S}^Y \quad (530)$$

then equation 525 can be simplified as

$$\mathbf{S}^Z = \mathbf{S}_{11}^X + \mathbf{T}_2 \mathbf{S}_{21}^X \quad (531)$$

and equation 527 can be simplified as

$$\mathbf{C}^Z = \mathbf{C}_1^X + \mathbf{T}_1 \mathbf{C}^Y + \mathbf{T}_2 \mathbf{C}_2^X. \quad (532)$$

Finally one can compute the noise correlation matrix  $\mathbf{N}^Z$  of the  $N$ -port  $\mathbf{Z}$  by calculating the correlation product of equation 532

$$\mathbf{N}^Z = \overline{\mathbf{C}^Z \mathbf{C}^{Z*}} = \mathbf{N}_{11}^X + \mathbf{N}_{12}^X \mathbf{T}_2^* + \mathbf{T}_2 \mathbf{N}_{21}^X + \mathbf{T}_2 \mathbf{N}_{22}^X \mathbf{T}_2^* + \mathbf{T}_1 \mathbf{N}^Y \mathbf{T}_1^* \quad (533)$$

with

$$\mathbf{N}^X = \overline{\mathbf{C}^X \mathbf{C}^{X*}} = \begin{bmatrix} \overline{\mathbf{C}_1^X \mathbf{C}_1^{X*}} & \overline{\mathbf{C}_1^X \mathbf{C}_2^{X*}} \\ \overline{\mathbf{C}_2^X \mathbf{C}_1^{X*}} & \overline{\mathbf{C}_2^X \mathbf{C}_2^{X*}} \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{11}^X & \mathbf{N}_{12}^X \\ \mathbf{N}_{21}^X & \mathbf{N}_{22}^X \end{bmatrix} \quad (534)$$

and

$$\mathbf{N}^Y = \overline{\mathbf{C}^Y \mathbf{C}^{Y*}} \quad (535)$$

where  $\mathbf{N}^X$  and  $\mathbf{N}^Y$  are the noise correlation matrices of the  $2N$ -port  $\mathbf{X}$  and the  $N$ -port  $\mathbf{Y}$ . The operator  $*$  denotes the conjugate transpose. Note that there is no correlation assumed between the noise waves  $\mathbf{C}^X$  and  $\mathbf{C}^Y$  of the  $2N$ -port  $\mathbf{X}$  and the  $N$ -port  $\mathbf{Y}$ .

A simplified notation can be achieved by introducing a new operator

$$\underbrace{(\mathbf{S}^Z, \mathbf{N}^Z)}_{\mathbf{Z}} = \underbrace{(\mathbf{S}^X, \mathbf{N}^X)}_{\mathbf{X}} \diamond \underbrace{(\mathbf{S}^Y, \mathbf{N}^Y)}_{\mathbf{Y}}. \quad (536)$$



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### L.9.6 De-cascading of a 2N-Port from a N-Port

De-cascading of a  $2N$ -port from a  $N$ -port is based on cascading of a  $2N$ -port and a  $N$ -port, see section L.9.5. To find the reverse function for de-cascading S-parameters, equation 525 can be rewritten as

$$\mathbf{S}_{12}^{X^{-1}} (\mathbf{S}^Z - \mathbf{S}_{11}^X) \mathbf{S}_{21}^{X^{-1}} = (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} \mathbf{S}^Y. \quad (537)$$

Setting

$$\mathbf{T}_0 = \mathbf{S}_{12}^{X^{-1}} (\mathbf{S}^Z - \mathbf{S}_{11}^X) \mathbf{S}_{21}^{X^{-1}} \quad (538)$$

and rearranging equation 537 yields

$$\mathbf{S}^Y = \mathbf{T}_0 (\mathbf{I} + \mathbf{S}_{22}^X \mathbf{T}_0)^{-1}. \quad (539)$$

To find the reverse function for de-cascading noise correlation matrices, equation 533 can be rewritten as

$$\mathbf{N}^Y = \mathbf{T}_1^{-1} (\mathbf{N}^Z - \mathbf{N}_{11}^X - \mathbf{N}_{12}^X \mathbf{T}_2^* - \mathbf{T}_2 \mathbf{N}_{21}^X - \mathbf{T}_2 \mathbf{N}_{22}^X \mathbf{T}_2^*) (\mathbf{T}_1^*)^{-1} \quad (540)$$

with

$$\mathbf{T}_2 = \mathbf{S}_{12}^X (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} \mathbf{S}^Y = \mathbf{S}_{12}^X \mathbf{T}_0 \quad (541)$$

$$\mathbf{T}_1 = \mathbf{S}_{12}^X (\mathbf{I} - \mathbf{S}^Y \mathbf{S}_{22}^X)^{-1} = \mathbf{T}_2 \mathbf{S}^{Y^{-1}}. \quad (542)$$

A simplified notation can be achieved by introducing a new operator

$$\underbrace{(\mathbf{S}^Y, \mathbf{N}^Y)}_{\mathbf{Y}} = \underbrace{(\mathbf{S}^X, \mathbf{N}^X)}_{\mathbf{X}} \triangleleft \underbrace{(\mathbf{S}^Z, \mathbf{N}^Z)}_{\mathbf{Z}}. \quad (543)$$

### L.9.7 Cascading of a 2N-Port and a 2N-Port

Cascading of a  $2N$ -port and a  $2N$ -port is based on cascading of a  $2N$ -port and a  $N$ -port, see section L.9.5.

One can expand the S-parameters  $\mathbf{S}^X$  and the noise correlation matrix  $\mathbf{N}^X$  of the  $2N$ -port  $\mathbf{X}$  to a  $4N$ -port  $\mathbf{X0}$

$$\mathbf{S}^{X0} = \begin{bmatrix} \mathbf{S}_{11}^X & \mathbf{0} & \mathbf{S}_{12}^X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{S}_{21}^X & \mathbf{0} & \mathbf{S}_{22}^X & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (544)$$

$$\mathbf{N}^{X0} = \begin{bmatrix} \mathbf{N}_{11}^X & \mathbf{0} & \mathbf{N}_{12}^X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{N}_{21}^X & \mathbf{0} & \mathbf{N}_{22}^X & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (545)$$

where  $\mathbf{I}$  is the identity matrix. In a next step the operator  $\diamond$  introduced in equation 536 can be used

$$\underbrace{(\mathbf{S}^Z, \mathbf{N}^Z)}_{\mathbf{Z}} = \underbrace{(\mathbf{S}^{X0}, \mathbf{N}^{X0})}_{\mathbf{X0}} \diamond \underbrace{(\mathbf{S}^Y, \mathbf{N}^Y)}_{\mathbf{Y}}. \quad (546)$$

A simplified notation can be achieved by introducing a new operator

$$\underbrace{(\mathbf{S}^Z, \mathbf{N}^Z)}_{\mathbf{Z}} = \underbrace{(\mathbf{S}^X, \mathbf{N}^X)}_{\mathbf{X}} \diamond \diamond \underbrace{(\mathbf{S}^Y, \mathbf{N}^Y)}_{\mathbf{Y}}. \quad (547)$$



### L.9.8 De-cascading of a 2N-Port from the left side of a 2N-Port

De-cascading of a 2N-Port from the left side of a 2N-Port is based on cascading of a 2N-Port and a 2N-Port, see section L.9.7.

One can expand the S-parameters  $\mathbf{S}^X$  and the noise correlation matrix  $\mathbf{N}^X$  of the 2N-port  $\mathbf{X}$  to a 4N-port  $\mathbf{X0}$  using equations 544 and 545. In a next step the operator  $\triangleleft$  introduced in equation 543 can be used

$$\underbrace{(\mathbf{S}^Y, \mathbf{N}^Y)}_Y = \underbrace{(\mathbf{S}^{X0}, \mathbf{N}^{X0})}_{\mathbf{X0}} \triangleleft \underbrace{(\mathbf{S}^Z, \mathbf{N}^Z)}_Z. \quad (548)$$

A simplified notation can be achieved by introducing a new operator

$$\underbrace{(\mathbf{S}^Y, \mathbf{N}^Y)}_Y = \underbrace{(\mathbf{S}^X, \mathbf{N}^X)}_X \triangleleft\triangleleft \underbrace{(\mathbf{S}^Z, \mathbf{N}^Z)}_Z. \quad (549)$$

### L.9.9 De-cascading of a 2N-Port from the right side of a 2N-Port

De-cascading of a 2N-Port from the right side of a 2N-Port is based on de-cascading of a 2N-Port from the left side of a 2N-Port, see section L.9.8.

In a first step each row and column of all S-parameter and noise correlation matrices are reversed. E.g.:  $\mathbf{S}_{rev}^X$  is the row and column reversed matrix of  $\mathbf{S}^X$ . In a next step the operator  $\triangleleft\triangleleft$  introduced in equation 549 can be used

$$\underbrace{(\mathbf{S}_{rev}^X, \mathbf{N}_{rev}^X)}_{\mathbf{X}_{rev}} = \underbrace{(\mathbf{S}_{rev}^Y, \mathbf{N}_{rev}^Y)}_{\mathbf{Y}_{rev}} \triangleleft\triangleleft \underbrace{(\mathbf{S}_{rev}^Z, \mathbf{N}_{rev}^Z)}_{\mathbf{Z}_{rev}}. \quad (550)$$

A simplified notation can be achieved by introducing a new operator

$$\underbrace{(\mathbf{S}^X, \mathbf{N}^X)}_X = \underbrace{(\mathbf{S}^Z, \mathbf{N}^Z)}_Z \triangleright\triangleright \underbrace{(\mathbf{S}^Y, \mathbf{N}^Y)}_Y. \quad (551)$$



### L.10 Derivation of Noise Parameters

This section describes the derivation of noise parameters using a flow chart for the noise receiver where  $\Gamma_s$  is the reflection coefficient and  $c_s$  the noise wave of the source, e.g.: tuner.

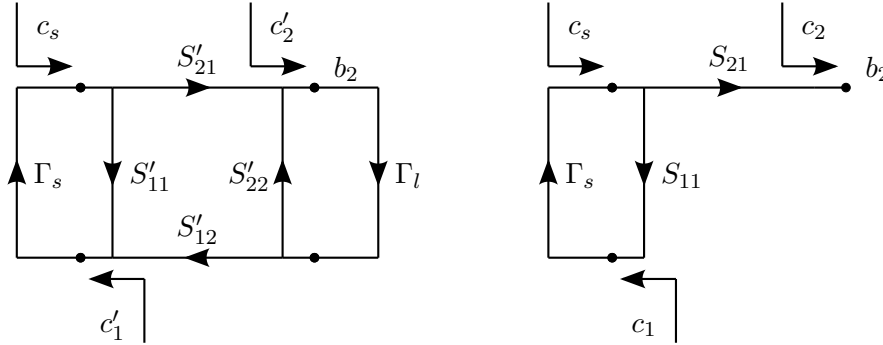


Figure 40: Noise receiver

$S'_{11}$ ,  $S'_{21}$ ,  $S'_{12}$  and  $S'_{22}$  are the S-parameters of the low noise amplifier (LNA).  $c'_1$  and  $c'_2$  are the noise waves of the LNA.  $\Gamma_l$  is the internal reflection coefficient of the noise receiver.

The S-parameters of the LNA and the internal reflection coefficient  $\Gamma_l$  of the noise receiver cannot not be measured directly. Only the cascaded S-parameters can be measured where  $S_{11}$  is the load match and  $S_{21}$  is the transmission tracking of the noise receiver.  $b_2$  is the sum of the received noise waves.

One can use the following two equations to calculate the load match  $S_{11}$  and the transmission tracking  $S_{21}$  of the noise receiver

$$S_{11} = S'_{11} + \frac{S'_{12}\Gamma_l S'_{21}}{1 - S'_{22}\Gamma_l} \quad (552)$$

$$S_{21} = \frac{S'_{21}}{1 - S'_{22}\Gamma_l}. \quad (553)$$

One can use the following two equations to compute the noise waves  $c_1$  and  $c_2$  of the noise receiver

$$c_1 = c'_1 + \frac{S'_{12}\Gamma_l}{1 - S'_{22}\Gamma_l} c'_2 \quad (554)$$

$$c_2 = \frac{1}{1 - S'_{22}\Gamma_l} c'_2. \quad (555)$$

The sum of the received noise waves is given by the following equation

$$b_2 = \frac{S_{21}}{1 - \Gamma_s S_{11}} c_s + \frac{\Gamma_s S_{21}}{1 - \Gamma_s S_{11}} c_1 + c_2. \quad (556)$$

One can compute the received noise power  $N^b$  in W/Hz by calculating the correlation product of the above equation

$$N^b = \overline{b_2 b_2^*} = \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} N^s + \frac{|\Gamma_s|^2 |S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} N_{11} + \frac{\Gamma_s S_{21}}{1 - \Gamma_s S_{11}} N_{12} + N_{21} \left( \frac{\Gamma_s S_{21}}{1 - \Gamma_s S_{11}} \right)^* + N_{22} \quad (557)$$



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with

$$N^s = \overline{c_s c_s^*} \quad (558)$$

and

$$\mathbf{N} = \begin{bmatrix} \overline{c_1 c_1^*} & \overline{c_1 c_2^*} \\ \overline{c_2 c_1^*} & \overline{c_2 c_2^*} \end{bmatrix} \quad (559)$$

where  $\mathbf{N}$  is the noise correlation matrix of the noise receiver. The operator  $*$  denotes the conjugate and the overbar the correlation product.

One can rearrange and simplify the above equation

$$N^b - \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} N^s = \frac{|\Gamma_s|^2 |S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \mathbf{N}_{11} + \mathbf{N}_{22} + 2 \operatorname{Re} \left( \frac{\Gamma_s S_{21}}{1 - \Gamma_s S_{11}} \mathbf{N}_{12} \right). \quad (560)$$

In a next step one can convert the noise parameter matrix  $\mathbf{N}$  of the above equation to NIST noise parameters  $X_1$ ,  $X_2$  and  $X_{12}$  using the following equation

$$\mathbf{N} = k \begin{bmatrix} X_1 & X_{12} S_{21}^* \\ X_{12}^* S_{21} & X_2 |S_{21}|^2 \end{bmatrix} \quad (561)$$

where  $k$  is the Boltzmann constant in J/K. Plugging equation 561 into equation 560 yields

$$\frac{N^b}{k} - \frac{|S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \frac{N^s}{k} = \frac{|\Gamma_s|^2 |S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} X_1 + |S_{21}|^2 X_2 + 2 \operatorname{Re} \left( \frac{\Gamma_s |S_{21}|^2}{1 - \Gamma_s S_{11}} X_{12} \right). \quad (562)$$

Finally one can divide the above equation by the available gain  $G_a$

$$G_a = \frac{(1 - |\Gamma_s|^2) |S_{21}|^2}{|1 - \Gamma_s S_{11}|^2} \quad (563)$$

which yields the effective input noise temperature  $T_e$  in kelvin

$$\begin{aligned} T_e &= \frac{1}{G_a} \frac{\overbrace{N^b}^{T_{ne}^b}}{k} - \frac{1}{\underbrace{|1 - |\Gamma_s|^2|}_{T_a^s}} \frac{\overbrace{N^s}^{T_{ne}^s}}{k} \\ &= \frac{|\Gamma_s|^2}{1 - |\Gamma_s|^2} X_1 + \frac{|1 - \Gamma_s S_{11}|^2}{1 - |\Gamma_s|^2} X_2 + \frac{2}{1 - |\Gamma_s|^2} \operatorname{Re} \left( (1 - \Gamma_s S_{11})^* \Gamma_s X_{12} \right) \end{aligned} \quad (564)$$

where  $T_{ne}^b$  is the measured effective noise temperature in kelvin.  $T_{ne}^s$  is the effective noise temperature in kelvin of the source.  $T_a^s$  is the available noise temperature of the source in kelvin which can be computed from the physical temperature  $T_p$  of the tuner, see equation 490.



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